37. On the Logarithm of Closed Linear Operators

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For a non-negative operator A in a Banach space X, Nollau [3] gave a definition of its logarithm $\log A$. In this note, we present another definition of $\log A$. Formally our definition is based on the relation

$$\log A = \log A(\mu+A)^{-1} - \log(\mu+A)^{-1}, \qquad \mu>0.$$

It is important here that $\log(\mu+A)^{-1}$ (resp. $\log A(\mu+A)^{-1}$) is to be defined as the infinitesimal generator of a holomorphic semi-group $(\mu+A)^{-\alpha}$, $\alpha\geqq 0$, (resp. $A^{\alpha}(\mu+A)^{-\alpha}$) under suitable conditions on A. Using this relation, we derive several formal properties of $\log A$, of which some seem to be new. By means of these properties, we finally give another proof of one of Nollau's representation formulas for $\log A$. The original proof was done through Dunford's integral and Nollau relied on this formula for the derivation of formal properties of $\log A$.

1. Definition and formal properties. We only consider a densely ranged and densely defined non-negative operator A in a Banach space X. Namely, all positive reals are contained in the resolvent set $\mathbf{P}(-A)$ of $-A$;

(1.1) $$\|r(r+A)^{-1}\|\leqq M, \qquad r>0;$$

(1.2) $$\overline{D(A)}=X;$$

(1.3) $$\overline{R(A)}=X.$$

Here $D(T)$, $R(T)$ stand for the domain and the range of an operator T, respectively. $\overline{Y}$ is the closure of the set Y in X.

For A with (1.1), (1.2), (1.3), the following assertion is well-known (Komatsu [1, 2], cf. Yosida [4]).

Proposition 1.1. *For any positive* μ, $\{(\mu+A)^{-\alpha};\alpha\geqq 0\}$, $\{A^{\alpha}(\mu+A)^{-\alpha};\alpha\geqq 0\}$ *are strongly continuous semi-groups of bounded linear operators. Both semi-groups are analytically continued to the half plane* $\mathrm{Re}\,\alpha>0$.

We also note the following relation (cf. Komatsu [2]):

(1.4) $$A^{\alpha}(\mu+A)^{-\alpha}=\mu^{-\alpha}(A^{-1}+\mu^{-1})^{-\alpha}.$$

We denote by $\Lambda^{+}(\mu;A)$ (resp. $\Lambda^{-}(\mu;A)$) the infinitesimal generator of $(\mu+A)^{-\alpha}$ (resp. $A^{\alpha}(\mu+A)^{-\alpha}$). We set $D^{\pm}(\mu;A)=D(\Lambda^{\pm}(\mu;A))$. We sometimes write $\Lambda^{\pm}(\mu)$, $D^{\pm}(\mu)$ instead of $\Lambda^{\pm}(\mu;A)$, $D^{\pm}(\mu;A)$.

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Proposition 1.2. *$D^+(\mu)$ and $D^-(\mu)$ are independent of $\mu>0$:*
$$D^+(\mu)=D^+,$$
$$D^-(\mu)=D^-.$$

Proof. We first note the following elementary

Lemma 1.3. *Let T_t be a strongly continuous group of bounded operators. If T_t is analytically continued in the sectors $\sum^\pm=\{\tau\,;\operatorname{Re}\tau\gtrless 0, |\arg\tau|<\theta_\pm\}$, $(0<\theta_\pm<\pi/2)$, then its infinitesimal generator is bounded, and T_t is entire in t.*

Proof. Under the assumption of Lemma, we see immediately that the spectrum of its infinitesimal generator B is compact. In particular,
$$T_t x=(2\pi i)^{-1}\int_\Gamma e^{-zt}(z-B)^{-1}x\,dz, \qquad (x\in X),$$
where Γ is a *bounded* closed curve containing the spectrum of B in its interior. This implies the lemma.

End of the proof of Proposition 1.2. If $\nu>0$, we have
$$(\mu+A)^{-\alpha}=\{(\nu+A)(\mu+A)^{-1}\}^\alpha(\nu+A)^{-\alpha}.$$
Since $\{(\nu+A)(\mu+A)^{-1}\}^\alpha$ is a group satisfying the hypothesis of Lemma 1.3, its infinitesimal generator $\Lambda_{\mu,\nu}$ is bounded. Thus, $x\in D^+(\nu)$ if and only if $x\in D^+(\mu)$, and
$$\Lambda^+(\mu)x=\Lambda^+(\nu)x+\Lambda_{\mu,\nu}x, \qquad x\in D^+.$$
The other half of Proposition 1.2 is proved similarly.

Proposition 1.4. *If $x\in D^+$ (resp. D^-), then $\Lambda^+(\mu)x$ (resp. $\Lambda^-(\mu)x$) is strongly differentiable in μ, and*

(1.5) $$d\Lambda^+(\mu)x/d\mu=-(\mu+A)^{-1}x, \qquad x\in D^+.$$

(resp.

(1.6) $$d\Lambda^-(\mu)x/d\mu=-(\mu+A)^{-1}x, \qquad x\in D^-).$$

Proof. Since
$$(\mu+A)^{-\alpha}x-(1+A)^{-\alpha}x=-\alpha\int_1^\mu(\nu+A)^{-\alpha-1}x\,d\nu,$$
we have, by differentiating in α and letting $\alpha\to 0$,
$$\Lambda^+(\mu)x-\Lambda^+(1)x=-\int_1^\mu(\nu+A)^{-1}x\,d\nu, \qquad x\in D^+.$$
Here we used that $\Lambda_{\mu,\nu}$ in the proof of Proposition 1.2 is continuous in μ or $\nu>0$. Hence, we get (1.5) after differentiation in μ. (1.6) is obtained analogously.

Corollary 1.5. *The operator L with*
$$Lx=\{\Lambda^-(\mu)-\Lambda^+(\mu)\}x, \qquad x\in D(L),$$
$$D(L)=D^+\cap D^-,$$
is defined independently of $\mu>0$.

Proposition 1.6. *The operator L is closable.*

Proof. First we note, for $x\in D^+\cap D^-$,

(1.7) $$(A^\alpha-I)(\mu+A)^{-\alpha}x=\int_0^\alpha A^\beta(\mu+A)^{-\alpha}Lx\,d\beta, \qquad \alpha>0,\ \mu>0.$$

In fact, if we set $u(\alpha)=(A^{\alpha}-I)(\mu+A)^{-\alpha}x$, $x\in D^{+}\cap D^{-}$, then we have

$$\begin{cases} du(\alpha)/d\alpha=\Lambda^{+}(\mu)u(\alpha)+A^{\alpha}(\mu+A)^{-\alpha}Lx, \\ u(0)=0. \end{cases}$$

Thus,

$$u(\alpha)=\int_0^{\alpha}\exp((\alpha-\beta)\Lambda^{+}(\mu))A^{\beta}(\mu+A)^{-\beta}Lxd\beta$$
$$=\int_0^{\alpha}A^{\beta}(\mu+A)^{-\alpha}Lxd\beta.$$

By differentiating (1.7) in μ, we obtain

(1.8) $$(A^{\alpha}-I)(\mu+A)^{-\alpha-1}x=\int_0^{\alpha}A^{\beta}(\mu+A)^{-\alpha-1}Lxd\beta, \qquad x\in D^{+}\cap D^{-}.$$

Now let $z_n\in D^{+}\cap D^{-}$ be such that $z_n\to z$, $Lz_n\to y$. From (1.8), we have

(1.9) $$(A^{\alpha}-I)(\mu+A)^{-\alpha-1}z=\int_0^{\alpha}A^{\beta}(\mu+A)^{-\alpha-1}yd\beta.$$

Here the right-hand side is differentiable in α, and $(\mu+A)^{-1}z\in D^{+}$. Thus the left-hand side of (1.9) is termwise differentiable in α. In particular, $(\mu+A)^{-1}z\in D^{-}$, and

$$\Lambda^{-}(\mu)(\mu+A)^{-1}z-\Lambda^{+}(\mu)(\mu+A)^{-1}z=(\mu+A)^{-1}y,$$

or

(1.10) $$L(\mu+A)^{-1}z=(\mu+A)^{-1}y.$$

Hence, if $z=0$, then $y=0$.

Corollary 1.7. *Let $L^{\sim}$ be the closure of L. If $x\in D(L^{\sim})$, then $(\mu+A)^{-1}x\in D(L)$, and*

$$L(\mu+A)^{-1}x=(\mu+A)^{-1}L^{\sim}x.$$

Proof. This follows from (1.10).

A similar discussion shows the following

Corollary 1.8. *If $x\in D(L^{\sim})$, then $A(\mu+A)^{-1}x\in D(L)$, and*

$$LA(\mu+A)^{-1}x=A(\mu+A)^{-1}L^{\sim}x.$$

Corollary 1.9. *If A is bounded, then $D(L)=D^{-}$, and $L^{\sim}=L$. If A^{-1} is bounded, then $D(L)=D^{+}$, and $L^{\sim}=L$.*

Proof. If A is bounded, then $(\mu+A)^{-\alpha}$ satisfies the conditions of Lemma 1.3. Thus, $D^{+}=X$. The other half follows similarly.

Definition 1.10. *We define* $\log A=L^{\sim}$.

Corollary 1.11. *If $x\in D^{+}\cap D^{-}$, then*

$$\log Ax=\log A(\mu+A)^{-1}x-\log(\mu+A)^{-1}x.$$

Proposition 1.12. $\log A=-\log A^{-1}$.

Proof. Using (1.4), we have

$$\Lambda^{+}(\mu;A)=-\log\mu+\Lambda^{-}(\mu^{-1};A^{-1}),$$
$$\Lambda^{-}(\mu;A)=-\log\mu+\Lambda^{+}(\mu^{-1};A^{-1}).$$

Proposition 1.13.

(1.11) $$\Lambda^{+}(\mu)=\log(\mu+A)^{-1};$$

(1.12) $$\Lambda^{-}(\mu)=\log A(\mu+A)^{-1}.$$

Proof. Put $B=(\mu+A)^{-1}$. Since

$$B^{\alpha}(1+B)^{-\alpha}x-(1+B)^{-\alpha}x$$
$$=(\mu+A)^{\alpha}(1+\mu+A)^{-\alpha}\int_0^{\alpha}\Lambda^+(\mu\,;A)(\mu+A)^{-\beta}xd\beta,$$

we have

$$\Lambda^-(1\,;B)B^{\alpha}(1+B)^{-\alpha}x-\Lambda^+(1\,;B)(1+B)^{-\alpha}x$$
$$=\Lambda^-(1\,;\mu+A)(\mu+A)^{\alpha}(1+\mu+A)^{-\alpha}\int_0^{\alpha}\Lambda^+(\mu\,;A)(\mu+A)^{-\beta}xd\beta$$
$$+(\mu+A)^{\alpha}(1+\mu+A)^{-\alpha}\Lambda^+(\mu\,;A)(\mu+A)^{-\alpha}x.$$

$\Lambda^+(1\,;B)=\Lambda^-(1\,;\mu+A)$ being bounded (Lemma 1.3), $x\in D(\Lambda^-(1\,;B))$ if and only if $x\in D(\Lambda^+(\mu\,;A))$, and

$$\Lambda^-(1\,;B)x-\Lambda^+(1\,;B)x=\Lambda^+(\mu\,;A)x.$$

By Corollary 1.9, we have (1.11). (1.12) is proved in a similar way.

2. Representation formula.

Proposition 2.1. *If* $x\in D(A^{\beta})$ *for some* $\beta, 0<\beta<1$, *then*

$$\lim_{R\to\infty}\log R(R+A)^{-1}x=0. \tag{2.1}$$

If $x\in R(A^{\beta'})$ *for some* $\beta', 0<\beta'<1$, *then*

$$\lim_{\varepsilon\to\infty}\log A(\varepsilon+A)^{-1}x=0. \tag{2.2}$$

Proof. Since

$$(R+A)^{-\alpha}x-R^{-\alpha}x=-\alpha\int_0^1 A(R+sA)^{-\alpha-1}xds, \qquad x\in D(A^{\beta}),$$
$$\Lambda^+(R\,;A)x-\log R^{-1}x=-\int_0^1 A(R+sA)^{-1}xds=0(R^{\beta-1}).$$

(2.2) is proved similarly.

Proposition 2.2. *If* $x\in D(A^{\beta})\cap R(A^{\beta'})$ *for some* $\beta,\beta', 0<\beta,\beta'<1$, *then*

$$\begin{aligned}\log Ax=(\log\nu)x+\lim_{R\to\infty}\int_{\nu}^{R}\mu^{-1}A(\mu+A)^{-1}xd\mu\\ -\lim_{\varepsilon\to\infty}\int_{\varepsilon}^{\nu}(\mu+A)^{-1}xd\mu\end{aligned} \tag{2.3}$$

for every $\nu>0$.

Proof. By Proposition 1.4,

$$d\log\mu(\mu+A)^{-1}x/d\mu=\{\mu^{-1}-(\mu+A)^{-1}\}x.$$

Thus, by Proposition 2.1,

$$-\log\nu(\nu+A)^{-1}x=\lim_{R\to\infty}\int_{\nu}^{R}\mu^{-1}A(\mu+A)^{-1}xd\mu.$$

Similarly,

$$\log A(\nu+A)^{-1}x=-\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\nu}(\mu+A)^{-1}xd\mu.$$

Hence, Corollary 1.11 implies the proposition.

Corollary 2.3 (Nollau). *If* $x\in D(A^{\beta})\cap R(A^{\beta'})$ *for some* $\beta,\beta', 0<\beta$, $\beta'<1$, *then*

$$\log Ax=\lim_{R\to\infty,\varepsilon\to 0}\int_{\varepsilon}^{R}(1+\mu)^{-1}(A-I)(\mu+A)^{-1}xd\mu. \tag{2.4}$$

Proof. We divide the integral in the right-hand side of (2.4):

$$\int_\varepsilon^R (1+\mu)^{-1}(A-I)(\mu+A)^{-1}x d\mu = \int_\varepsilon^\nu + \int_\nu^R = I_\varepsilon + I^R, \qquad \nu > 0.$$

Then, as easily seen,

$$I_\varepsilon = \{\log(\nu+1) - \log(1+\varepsilon)\}x - \int_\varepsilon^\nu (\mu+A)^{-1}x d\mu,$$

and

$$I^R = \int_\nu^R \mu^{-1}A(\mu+A)^{-1}x d\mu - \{\log R(R+1)^{-1} - \log\nu + \log(\nu+1)\}x.$$

Hence, (2.4) follows from (2.3).

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