92. *On the Index of Hypoelliptic Pseudo-differential Operators on R^n*

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§0. Introduction. The purpose of this paper is to prove that the index of a system P of pseudo-differential operators on R^n vanishes, if the symbol $\sigma(P)(x,\xi)$ is slowly varying in the sense of Grushin [1] and satisfies the condition which is a modification of Hörmander's condition for the existence of parametrix (cf. Hörmander [3] and Šubin [8]).

We shall denote by $S^m_{\rho,\delta}$, $0\leqq\delta<\rho\leqq 1$, $-\infty<m<\infty$, the set of all C^∞-symbols $p(x,\xi)$ defined in $R^n_x\times R^n_\xi$, which satisfy for multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$,

(0.1) $$|p^{(\beta)}_{(\alpha)}(x,\xi)|\leqq C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for some constants $C_{\alpha,\beta}$, where $p^{(\beta)}_{(\alpha)}(x,\xi)=D^\alpha_x\partial^\beta_\xi p(x,\xi)$, $D^\alpha_x=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}$, $\partial^\beta_\xi=(\partial/\partial\xi_1)^{\beta_1}\cdots(\partial/\partial\xi_n)^{\beta_n}$. We set $S^\infty_{\rho,\delta}=\bigcup_m S^m_{\rho,\delta}$ and $S^{-\infty}_{\rho,\delta}=\bigcap_m S^m_{\rho,\delta}$. For a symbol $p(x,\xi)\in S^m_{\rho,\delta}$ we define a pseudo-differential operator $p(X,D_x)$ by

(0.2) $$p(X,D_x)u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}(\xi)$ denotes the Fourier transform of a rapidly decreasing function $u(x)$ defined by $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$.

We say that a symbol $p(x,\xi)$ $(\in S^m_{\rho,\delta})$ is slowly varying, if the estimate (0.1) holds for a bounded function $C_{\alpha,\beta}(x)$ such that $C_{\alpha,\beta}(x)\to 0$ as $|x|\to\infty$ in case $\alpha\neq 0$ (cf. Grushin [1], p. 206).

Our main theorem is stated as follows:

Main theorem. *Let $p(x,\xi)=(p_{jk}(x,\xi))$ be an $l\times l$ matrix of symbols $p_{jk}(x,\xi)$ of class $S^m_{\rho,\delta}$, $m>0$, which are slowly varying. Assume that there exist positive constants c_0, c_1 and $0<\tau\leqq 1$ such that $(p(x,\xi)-\zeta I)^{-1}$ exists on*

$$\Xi_\xi=\{\zeta\in \boldsymbol{C};\ \mathrm{dis}\,(\zeta,(-\infty,0])\leqq c_0(1+|\xi|)^{\tau m}\}$$

and the estimate of the form

(0.3) $$\|p^{(\beta)}_{(\alpha)}(x,\xi)(p(x,\xi)-\zeta I)^{-1}\|\leqq C_{\alpha,\beta}(x)(1+|\xi|)^{\delta|\alpha|-\rho|\beta|}$$

holds uniformly on Ξ_ξ, where $\|\cdot\|$ denotes a matrix norm, and $C_{\alpha,\beta}(x)$ is a bounded function which tends to zero as $|x|\to\infty$ in case $\alpha\neq 0$. Then the operator $P=p(X,D_x)$ as the map from $L^2=L^2(R^n)$ into itself with the domain $\mathcal{D}(P)=\{u\in L^2;\ Pu\in L^2\}$ is Fredholm type and we have

(0.4) $$\text{index } P \equiv \dim\ker P - \operatorname{codim} \mathcal{R}_e P = 0.$$

Remark. We may assume the condition (0.3) only in $\{(x,\xi);\ |\xi| \geqq \gamma(x)\}$ for a continuous function $\gamma(x)$ ($\geqq 0$) with compact support. The reason will be explained after the proof of main theorem in § 2. For example, the symbol

$$p(x,\xi) = |\xi|^2 + \{|x|^2(1+|x|^2)^{-1} - n(1+|x|^2)^{-1/2} + |x|^2(1+|x|^2)^{-3/2}\}$$

satisfies (0.3) in $\{(x,\xi);\ |\xi| \geqq \gamma(x)\}$ for a continuous function $\gamma(x)$ with compact support such that $\gamma(x) > 0$ in $\{x;\ |x| \leqq R\}$ for a large $R > 0$, and is slowly varying. Then we have "index $p(X, D_x) = 0$", but the dimension of the null space of $p(X, D_x)$ does not vanish, since

$$p(X, D_x) \exp(-\sqrt{1+|x|^2}) = 0.$$

For the proof we shall apply Proposition 2.1 in Hörmander [4] to a one parameter family $\{P_z\}_{z \in C}$ of complex powers P_z for P which has been constructed in Kumano-go and Tsutsumi [6], and shows "index P = index P_z ($\mathfrak{Re}\, z \geqq 0$) = index $I = 0$". In the present paper only the sketch of the proof is given and the detailed description will be given there.

§ 1. Preliminary lemmas. Let H_s, $-\infty < s < \infty$, be the Sobolev space with the s-norm $\|u\|_s^2 = (2\pi)^{-n}\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$.

Proposition 1.1. *Let $p_j(x,\xi)$, $j = 0, 1, 2, \cdots$, be a sequence of slowly varying symbols of class $S^{m_j}_{\rho,\delta}$ such that $m_j \downarrow -\infty$ as $j \to \infty$. Then we can construct a slowly varying symbol $p(x,\xi) \in S^{m_0}_{\rho,\delta}$ such that*

(1.1) $$p(x,\xi) - \sum_{j=1}^{N-1} p_j(x,\xi) \in S^{m_N}_{\rho,\delta}$$

and is slowly varying for any N (cf. [3]).

Proof. Take C^∞-functions $\varphi(\xi)$ and $\psi(x,\xi)$ such that $\varphi(\xi) = 0$ ($|\xi| \leqq 1$), $= 1$ ($|\xi| \geqq 2$) and $\psi(x,\xi) = 0$ ($|x| + |\xi| \leqq 1$), $= 1$ ($|x| + |\xi| \geqq 2$). Then, setting $p(x,\xi) = p_0(x,\xi) + \sum_{j=1}^\infty \varphi(t_j^{-1}\xi)\psi(t_j^{-1}x, t_j^{-1}\xi) p_j(x,\xi)$ for an appropriate $t_j \uparrow \infty$ ($j \to \infty$), we get a required symbol. Q.E.D.

Let $P = (p_{jk}(X, D_x))$ be an $l \times l$ matrix of pseudo-differential operators $p_{jk}(X, D_x)$ of class $S^m_{\rho,\delta}$ and $\tilde{P}^*$ be the formal adjoint of P in the sense: $(Pu, v) = (u, \tilde{P}^* v)$ for $u, v \in H_\infty (= \bigcap_s H_s)$. Then, by Kumano-go [5], p. 33 and 43, $\tilde{P}^*$ belongs to $S^m_{\rho,\delta}$ and whose symbol $\sigma(\tilde{P}^*)(x,\xi) = (\tilde{p}^*_{jk}(x,\xi))$ is defined by

(1.2) $$\tilde{p}^*_{jk}(x,\xi) = (2\pi)^{-n}\int\left\{\int e^{-iw\cdot\zeta}\langle w\rangle^{-n_0}\langle D_\zeta\rangle^{n_0}\overline{p_{kj}(x+w, \xi+\zeta)}\,dw\right\}d\zeta$$

($n_0 \geqq n+1$, any even number).

Taking large n_0 we can see that $\tilde{p}^*_{jk}(x,\xi)$ are slowly varying, if $p_{jk}(x,\xi)$ are so (cf. Grushin [1]).

Proposition 1.2. *Let P be an $l \times l$ matrix of pseudo-differential operators of class $S^m_{\rho,\delta}$, and let $\tilde{P}^*$ be the formal adjoint. Set*

(1.3) $$\mathcal{D}(P) = \{u \in L^2;\ Pu \in L^2\}, \qquad \mathcal{D}(\tilde{P}^*) = \{v \in L^2;\ \tilde{P}^* v \in L^2\}.$$

Then for the adjoint operator P^* of P we have $P^* \subset \tilde{P}^*$.

Proof. Let $v \in \mathcal{D}(P^*)$. Since $H_\infty \subset \mathcal{D}(P)$, we have $(u, \tilde{P}^*v) = (Pu, v) = (u, P^*v)$ for $u \in H_\infty$. Then, we have $\tilde{P}^*v = P^*v \in L^2$ which proves $P^* \subset \tilde{P}^*$. Q.E.D.

Now for $p(x, \xi) \in S^m_{\rho,\delta}$ we introduce semi-norms

$$|p|_{m,k} = \underset{|\alpha+\beta|\leqq k}{\mathrm{Max}} \sup_{x,\xi} \{|p^{(\beta)}_{(\alpha)}(x, \xi)|(1+|\xi|)^{-(m+\delta|\alpha|-\rho|\beta|)}\} \quad (k=0,1,2,\cdots). \tag{1.4}$$

Then $S^m_{\rho,\delta}$ makes a Fréchet space, and for any $p(x,\xi) \in S^0_{\rho,\delta}$ we have

$$\|p(X, D_x)u\|_0 \leqq C_0 |p|_{0,k_0} \|u\|_0, \qquad u \in L^2, \tag{1.5}$$

where C_0 and k_0 are constants independent of $p(x, \xi)$ (cf. [5], p. 48).

Lemma 1.3. *Let $\{p_s(x, \xi)\}_{-s_0 \leqq s \leqq 0}$ $(s_0 > 0)$ makes a bounded set in $S^0_{\rho,\delta}$. Assume that*

$$p_s(x, \xi) \to p_0(x, \xi) \quad on \quad R^n_x \times K \ (s \uparrow 0) \tag{1.6}$$

uniformly for any compact set K of R^n_ξ. Then we have for any $u \in L^2$

$$p_s(x, D_x)u \to p_0(X, D_x)u \quad \text{in} \quad L^2\ (s \uparrow 0). \tag{1.7}$$

Proof. Take a C^∞-symbol $\Phi(\xi)$ such that $0 \leqq \Phi(\xi) \leqq 1$, $\Phi(\xi) = 1$ $(|\xi| \leqq 1)$, $=0 (|\xi| \geqq 2)$, and set $\Phi_\varepsilon(\xi) = \Phi(\varepsilon\xi)$. Then $\{\Phi_\varepsilon(\xi)\}_{0<\varepsilon\leqq 1}$ makes a bounded set in $S^0_{1,0} \subset S^0_{\rho,\delta}$, and for any fixed $\varepsilon_0 > 0$

$$(p_s(x, \xi) - p_0(x, \xi))\Phi_{\varepsilon_0}(\xi) \to 0 \quad \text{in} \quad S^0_{\rho,\delta}. \tag{1.8}$$

Setting $P_s = p_s(X, D_x)$ we write

$$(P_s - P_0)u = (P_s - P_0)\Phi_\varepsilon(D_x)u + (P_s - P_0)(1 - \Phi_\varepsilon(D_x))u.$$

Then by (1.5) we have

$$\begin{aligned} &\|(P_s - P_0)(1-\Phi_\varepsilon(D_x))u\|_0^2 \leqq C\|(1-\Phi_\varepsilon(D_x))u\|_0^2 \\ &\leqq C' \int |(1-\Phi_\varepsilon(\xi))\hat{u}(\xi)|^2 \, d\xi \leqq C' \int_{|\xi|\geqq \varepsilon^{-1}} |\hat{u}(\xi)|^2 d\xi \to 0 (\varepsilon \downarrow 0). \end{aligned} \tag{1.9}$$

On the other hand for a fixed $\varepsilon_0 > 0$ we have by (1.5) and (1.8)

$$\|(P_s - P_0)\Phi_{\varepsilon_0}(D_x)u\|_0 \leqq C_0 |(P_s - P_0)\Phi_{\varepsilon_0}|_{0,k_0} \|u\|_0 \to 0 (s \uparrow 0). \tag{1.10}$$

From (1.9) and (1.10) we get (1.7). Q.E.D.

Lemma 1.4. *Let $p_z(x, \xi)$, $z \in \Omega$, be a subset of $S^0_{\rho,\delta}$ and an analytic function of z in the topology of $S^0_{\rho,\delta}$, where Ω is an open set of $\boldsymbol{C}$. Then $p_z(X, D_x)u$ is an analytic function of z in the L^2-topology.*

Proof is omitted.

Definition 1.5. For an $l \times l$ matrix $P = p(X, D_x) \in S^m_{\rho,\delta}$, $m > 0$, we say that $\{P_z\}_{z \in \boldsymbol{C}} (\subset S^\infty_{\rho,\delta})$ is a one parameter family of complex powers P_z for P, if $\{P_z\}_{z\in\boldsymbol{C}}$ satisfies the following:

i) For a monotone increasing function $m(s)$ such that $m(s) \to -\infty$ $(s \to -\infty)$, $m(0) = 0$, $m(s) \to \infty (s \to \infty)$ we have $P_z \in S^{m(\mathfrak{Re}\, z)}_{\rho,\delta}$.

ii) $P_0 = I$ (identity operator), $P_1 = P$ (original operator).

iii) For any real s_0 $\sigma(P_z)(x, \xi)$ is an analytic function of z $(\mathfrak{Re}\, z < s_0)$ in the topology of $S^{m(s_0)}_{\rho,\delta}$.

vi) For any $s_0 > 0$ $\{\sigma(P_s)(x, \xi)\}_{-s_0 \leqq s \leqq 0}$ makes a bounded set in $S^0_{\rho,\delta}$ and

$$\sigma(P_s)(x, \xi) \to I \quad \text{on} \quad R^n_x \times K \ (s \uparrow 0) \tag{1.11}$$

uniformly for any compact set K of R^n_ξ.

v) $P_{z_1}P_{z_2}-P_{z_1+z_2}\in S^{-\infty}_{\rho,\delta}$ for any $z_1, z_2 \in C$ in the sense:

$$\sigma(P_{z_1}P_{z_2}-P_{z_1+z_2})(x,\xi)$$

is an analytic functions of z_1 and z_2 in the topology of $S^{s_0}_{\rho,\delta}$ for any real s_0.

Then we have

Theorem 1.6. *Let $P\in S^m_{\rho,\delta}$ for $m>0$. Assume that there exists a one parameter family $\{P_z\}_{z\in C}$ of complex powers for P. Then we have, for any $z_0\in C$, $\tilde{P}^*_{z_0}=P^*_{z_0}$ as the map of L^2 into itself.*

Proof. By Proposition 1.2 we have only to prove

$$(P_{z_0}u, v)=(u, \tilde{P}^*_{z_0}v) \quad \text{for} \quad u\in\mathcal{D}(P_{z_0}), \quad v\in\mathcal{D}(\tilde{P}^*_{z_0}). \tag{1.12}$$

By i) we have $P_z u\in H_{-m(\Re z)}$ for $u\in\mathcal{D}(P_{z_0})$, so we have for a large N

$$\begin{aligned}(P_z u, \tilde{P}^*_{z_0}v)&=(P_{z_0}P_z u, v)=(P_z P_{z_0}u, v)\\ &+((P_{z_0}P_z-P_zP_{z_0})u, v), \quad u\in\mathcal{D}(P_{z_0}), \quad v\in\mathcal{D}(\tilde{P}^*_{z_0})\\ &\qquad (\Re z<-N).\end{aligned} \tag{1.13}$$

From Lemmas 1.3–1.4 and iii)-iv) we have $(P_z u, \tilde{P}^*_{z_0}v)$ is analytic in z when $\Re z<0$ and $\lim_{s\to -0}(P_s u, \tilde{P}^*_{z_0}v)=(u, \tilde{P}^*_{z_0}v)$. Since $P_{z_0}u\in L^2$, we also have that $(P_zP_{z_0}u, v)$ is analytic in z when $\Re z<0$ and $\lim_{s\to -0}(P_sP_{z_0}u, v)=(P_{z_0}u, v)$. Setting $s_0=0$ in v) and writing

$$P_{z_0}P_z-P_zP_{z_0}=(P_{z_0}P_z-P_{z_0+z})+(P_{z_0+z}-P_{z_0}P_z),$$

we can see that $((P_{z_0}P_z-P_zP_{z_0})u, v)$ is analytic in z and

$$\lim_{s\to -0}((P_{z_0}P_s-P_sP_{z_0})u, v)=0.$$

Then letting $z\to -0$ on the real line in (1.13), we get (1.12). This completes the proof. Q.E.D.

§2. Proof of main theorem.

We exhibit a theorem given in [6].

Theorem 2.1. *Let $P=p(X, D_x)$ be an $l\times l$ matrix of pseudo-differential operators whose symbols satisfy the condition of main theorem. Then we can construct a one parameter family $\{P_z\}_{z\in C}$ of complex powers for P such that $m(s)=\tau ms$ for $s<0$, $=ms$ for $s\geqq 0$, $\sigma(P_z)(x,\xi)$ are slowly varying, and $P_{z_1}P_{z_2}-P_{z_1+z_2}$ are compact from H_{s_1} into H_{s_2} for any real s_1, s_2.*

Sketch of the proof. We first construct a parametrix $r(\zeta; X, D_x)=\sum_{j=0}^{\infty}q_j(\zeta; X, D_x)\in S^{-\tau m}_{\rho,\delta}$ of $(P-\zeta I)$ in $\Xi_0=\{\zeta\in C;\ \mathrm{dis}\,(\zeta, (-\infty, 0])\leqq c_0\}$ by the usual way. Then we note that $q_0(\zeta; x,\xi)=(P(x,\xi)-\zeta I)^{-1}$ and, for $q_j(\zeta; x,\xi)\ (\in S^{-\tau m-j(\rho-\delta)}_{\rho,\delta})$, $j=1,2,\cdots$, we can take bounded functions $C_{\alpha,\beta}(x)$ as $C_{\alpha,\beta}$ of (0.1) such that $C_{\alpha,\beta}(x)\to 0(|x|\to\infty)$ for any α,β. This fact derives the last statement of Theorem 2.1. According to Hayakawa and Kumano-go [2] consider the Dunford integral

$$p_z(x,\xi)=(2\pi i)^{-1}\int_\Gamma \zeta^z r(\zeta; x,\xi)d\zeta \qquad (\Re z<0), \tag{2.1}$$

where Γ is an oriented curve defined by $\{\zeta\in C;\ \mathrm{dis}\,(\zeta,(-\infty,0])\leqq c_0/2\}$. Then $p_z(x,\xi)$ has the analytic continuation to the whole z-plane. We can see also

$$(2.2)\quad \int_\Gamma \zeta^z(p(x,\xi)-\zeta I)^{-1}d\zeta = p(x,\xi)^z, \quad \int_\Gamma \zeta^z q_j(\zeta\,;x,\xi)d\zeta=0, \quad j=1,2,\cdots \quad (z=0,1),$$

so that we have $p_0(x,\xi)=1$ and $p_1(x,\xi)=p(x,\xi)$. For the group property and the continuity in the $S^s_{\rho,\delta}$-topology we have to do careful calculation which is omitted here. For the scalar case we can see the concrete form in Nagase and Shinkai [7].

We use Proposition 2.1 of Hörmander [4] in the following modified form:

Lemma 2.2. *Let $\{P_t\}_{0\leqq t\leqq 1}$ be the family of an $l\times l$ matrix of pseudo-differential operators of class $S^{s_0}_{\rho,\delta}$ for some s_0 such that $\tilde{P}^*_t=P^*_t$ as the operators of L^2 into itself. Suppose that $\sigma(P_t)(x,\xi)$ and $\sigma(\tilde{P}^*_t)(x,\xi)$ are continuous in the $S^{s_0}_{\rho,\delta}$-topology, and that there exists a family $\{Q\}_{0\leqq t\leqq 1}$ of parametrices $Q_t(\in S^0_{\rho,\delta})$ for P_t such that Q_t is strongly continuous in the L^2-topology, $\{\sigma(Q_t)(x,\xi)\}_{0\leqq t\leqq 1}$ is bounded in $S^0_{\rho,\delta}$ and*

$$(2.3)\qquad Q_tP_t\subset I+K_t, \qquad P_tQ_t\subset I+K'_t,$$

where K_t and K'_t are compact operators in L^2 and continuous in the uniform L^2-topology. Then P_t is Fredholm type and

$$(2.4)\qquad \text{index}\, P_t=\dim\ker P_t-\text{codim}\,\mathcal{R}_e\, P_t=\text{Const.} \qquad \text{on } [0,1].$$

Proof is omitted, but we only note that the graphs:

$$G=\{(t,u,P_tu)\,;\,u\in\mathcal{D}(P_t),\,t\in[0,1]\}$$

and

$$G^*=\{(t,v,\tilde{P}^*_tv)\,;\,v\in\mathcal{D}(\tilde{P}^*_t),\,t\in[0,1]\}$$

are closed in $[0,1]\times L^2\times L^2$.

Proof of main theorem. By means of Theorem 2.1 we have a one parameter family $\{P_z\}_{z\in C}$ of complex powers P_z for P. We consider $\{P_t\}_{0\leqq t\leqq 1}$. Then we have P_{-t} as Q_t in Lemma 2.2. Then we have by (2.4) "index $P=$ index $P_1=$ index $P_0=$ index $I=0$". Q.E.D.

Remark for main theorem. When $p(x,\xi)$ satisfies the condition (0.3) in $\{(x,\xi)\,;\,|\xi|\geqq\gamma(x)\}$ we take a C_0^∞-function $a(x)(\geqq 0)$ which is equal to 1 on supp γ. Then $p_0(x,\xi)=p(x,\xi)+\lambda_0a(x)I$ satisfies (0.3) in $R^n_x\times R^n_\xi$ for a fixed $\lambda_0>0$. Hence we have index $p_0(X,D_x)=0$. Set $P_t=p_0(X,D_x)-t\lambda_0a(x)I$, $0\leqq t\leqq 1$, and $Q_t=Q$ where $Q(\in S^{-\tau m}_{\rho,\delta})$ is the parametrix for $p_0(X,D_x)$. Then $P_0=p_0(X,D_x)$, $P_1=p(X,D_x)$, and, noting that $Qa(x)$ and $a(x)Q$ are compact, we see that we can apply Lemma 2.2 to $\{P_t\}_{0\leqq t\leqq 1}$. Hence $p(X,D_x)$ is Fredholm type and "index $p(X,D_x)=$ index $p_0(X,D_x)=0$".

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