41. *On Some Non-linear Equations*

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1. Problems and results. In this note we study the following equations:

$$(1.1)\qquad \begin{cases} \displaystyle\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x,u)\frac{\partial u}{\partial x_k}\right) - \lambda u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & (n \leqq 3), \end{cases}$$

and

$$\frac{\partial u}{\partial t} = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x,u)\frac{\partial u}{\partial x_k}\right), \qquad t>0,\ x\in\Omega$$

$$u(x,t)|_{\partial\Omega} = 0,\ u(x,0) = u_0(x) \qquad (n \leqq 3),$$

where Ω is a bounded smooth domain in R^n.

We assume here

(A.1) $a_{jk}(x,y) = a_{kj}(x,y)$ are real-valued and of class $\mathscr{B}^1(\bar{\Omega}\times R^1)$.

(A.2) There exists a positive constant c such that

$$\sum_{j,k=1}^{n} a_{jk}(x,y)\xi_j\xi_k \geqq c\,|\xi|^2$$

for any $\xi = (\xi_1, \cdots, \xi_n) \in R^n$ and any $(x,y) \in \bar{\Omega}\times R^1$.

As for elliptic equations there are many works on more general ones than (1.1), and most of them are concerned with 'weak' solutions such as belonging to $H^1(\Omega)$, or treated in a space of Hölder class. One of our aims here is to show the existence of 'strong' solutions of (1.1), which belong to $H^2(\Omega)$, though the dimension of the underlying domain is restricted. Another is to show, if the initial data is small enough, then there exists a unique global solution of (1.2) belonging to $H^2(\Omega)$ for each t, and it will give an example which makes it possible to apply the abstract theory on non-linear semi-groups to some quasi-linear parabolic equations.

Our results are

Theorem 1. *For any positive number a, there exists a real number $\lambda_0 = \lambda_0(a)$ such that the equation* (1.1) *has a solution $u(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $\lambda \geqq \lambda_0(a)$ and $f(x) \in L^2(\Omega)$ with $\|f\| \leqq a$.*

Theorem 2. *In Theorem* 1, *we can take $\lambda_0(a) = 0$ for sufficiently small a.*

Theorem 3. *If $\|u_0\|_2$ is sufficiently small, then the equation* (1.2) *has a unique solution $u(t,\cdot) \in H^2(\Omega)\cap H_0^1(\Omega)$ $(t \geqq 0)$ such that:*

(i) *$u(t,\cdot)$ is Lipschitz continuous function in $L^2(\Omega)$, so that there exists $du/dt \in L^2(\Omega)$ for almost all $t>0$, which satisfies* (1.2).

(ii) *The right derivative* $D_t^+u(t, \cdot) \in L^2(\Omega)$ *exists for all* $t \geqq 0$ *and it satisfies* (1.2).

2. Preliminaries. In this section we shall prepare some lemmas and propositions which will be needed to prove Theorems 1 and 2.

We start with

Lemma 2.1. *Let* $u(x)$ *be in* $H^2(\Omega)$ *and* $v(x)$ *be in* $H^s(\Omega)$ $(s>n/2)$. *Then it follows in the sense of* $L^2(\Omega)$ *that*

$$\text{(2.1)} \quad \frac{\partial}{\partial x_j}\left(a_{jk}(x,v)\frac{\partial u}{\partial x_k}\right) = a_{jk,j}(x,v)\frac{\partial u}{\partial x_k} + a'_{jk}(x,v)\frac{\partial v}{\partial x_j}\frac{\partial u}{\partial x_k} + a_{jk}(x,v)\frac{\partial^2 u}{\partial x_j \partial x_k}$$

where $a_{jk,j}$ *denotes the partial derivatives of* $a_{jk}(x,y)$ *by* x_j *and* a'_{jk} *by* y.

In developing our arguments, the following well-known lemma due to Sobolev is much important.

Lemma 2.2 (Sobolev). *Let* $u(x)$ *be in* $H^s(\Omega)$. *If* $0<s<n/2$, *then* $u(x)$ *is in* $L^p(\Omega)$ *with* $1/p=1/2-s/n$, *and if* $s>n/2$, *then* $u(x)$ *is in* $\mathscr{B}^\sigma(\Omega)$ *with* $0<\sigma<s-n/2$, $\sigma \leqq 1$. *And in both cases, the imbedding is bounded.*

The following *a priori* estimate also plays an important rôle.

Lemma 2.3. *Let* $u(x)$ *be in* $H^2(\Omega) \cap H_0^1(\Omega)$ *and* $v(x)$ *be in* $H^s(\Omega)$ *with* $7/4<s\leqq 2$, *and put*

$$\text{(2.2)} \qquad \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x,v)\frac{\partial u}{\partial x_k}\right) = f(x),$$

which is in $L^2(\Omega)$ *by Lemma* 2.1. *Then we have*

$$\text{(2.3)} \qquad \|u\|_2 \leqq \text{const.}\ \{\|f\| + (1+\|v\|_s)^3\|u\| + (1+\|v\|_s)^2\|u\|_1\}.$$

The proof of this lemma can be carried out by localization by partition of unity as well as the case of linear equations, and the basic techniques are almost similar to the linear cases except that the diameter of localization depends upon $\|v\|_s$.

The following two propositions are prepared for applying the Schauder's fixed point theorem.

Proposition 2.1. *Let* $u(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $v(x) \in H^s(\Omega)$ *satisfy*

$$\text{(2.4)} \qquad \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x,v)\frac{\partial u}{\partial x_k}\right) - \lambda u = f, \quad f \in L^2(\Omega), \quad \lambda \geqq 0.$$

Then we have

$$\text{(2.5)} \qquad \|u\|_2 \leqq \text{const.}\ [\{(1+\|v\|_s)^5 + \lambda\}(\lambda+\gamma)^{-1} + 1]\,\|f\|.$$

$$\text{(2.6)} \qquad \|u\| \leqq (\lambda+\gamma)^{-1}\|f\|,$$

for some constant $\gamma>0$.

Proposition 2.2. *Let* $v_j(x) \in H^s(\Omega)$ $(7/4<s\leqq 2)$ *and* $u_j(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ *satisfy* (2.4) *for* $j=1,2$, *and* $\|v_j\|_s \leqq M$. *Then for any* $\varepsilon>0$, *there exists* $\delta=\delta(\varepsilon, M)>0$, *such that* $\|v_1-v_2\|_s<\delta$ *implies* $\|u_1-u_2\|_2<\varepsilon$.

3. Sketch of Proof of Theorems 1 and 2. The proof of Theorems 1 and 2 is nothing but a direct application of the Schauder's fixed point theorem. For this we describe a few lemmas.

Lemma 3.1. *For any* $f(x) \in L^2(\Omega)$ *and* $v \in H^s(\Omega)$ $(7/4 < s \leqq 2)$, *the equation* (2.4) *has a unique solution* $u(x) \in H^2(\Omega) \cap H_0^1(\Omega)$.

This is an immediate consequence of Proposition 2.2.

Thus we have a mapping $T = T_{\lambda, f}: v \to u$ from $H^2(\Omega) \cap H_0^1(\Omega)$ into itself for each $\lambda \geqq 0$ and $f \in L^2(\Omega)$. Since the imbedding $H^2(\Omega) \cap H_0^1(\Omega)$ into $H^s(\Omega) \cap H_0^1(\Omega)$ $(s < 2)$ is compact, by Proposition 2.2 again, we have

Lemma 3.2. *The mapping T is compact.*

It is easy to show by Proposition 2.1 that

Lemma 3.3. *Let* $a > 0$. *Then there exists* $\lambda_0 = \lambda_0(a)$ *and* $L = L(\lambda, a)$ *for* $\lambda \geqq \lambda_0(a)$ *such that* $T_{\lambda, f}$ *maps*

$$D_{\lambda, a} \equiv \{u \in H^2(\Omega) \cap H_0^1(\Omega)\,;\, \|u\| \leqq a(\lambda + \gamma)^{-1},\, \|u\|_2 \leqq L(\lambda, a)\}$$

into itself if $\|f\| \leqq a$. $\lambda_0(a) = 0$ *for sufficiently small* a.

By virtue of Lemma 3.2 and Lemma 3.3, we can apply the Schauder's fixed point theorem, which yields Theorems 1 and 2.

4. Proof of Theorem 3. The arguments to prove Theorem 2 shows that

Lemma 4.1. *Let* $a > 0$ *be sufficiently small. Then for any* $f \in \mathscr{B}_a \equiv \{f \in L^2(\Omega)\,;\, \|f\| < a\}$, *the equation*

$$A(u) \equiv \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x, u)\frac{\partial u}{\partial x_k}\right) = f$$

$$u|_{\partial\Omega} = 0$$

has a solution $u(x)$ *in*

$$D_a \equiv \{u \in H^2(\Omega) \cap H_0^1(\Omega)\,;\, \|u\| \leqq a\gamma^{-1},\, \|u\|_2 \leqq L(0, a)\},$$

where $L(0, a)$ *tends monotonously to* 0 *as* a *tends to* 0.

As for the monotonicity of the operator A, we have

Lemma 4.2. *For any* u *and* v *in* D_a, *it holds that*

$$(A(u) - A(v), u - v) \leqq -\beta \|u - v\|^2, \tag{4.1}$$

with some positive constant β *if* $a > 0$ *is sufficiently small.*

Now let a be so small as Lemmas 4.1 and 4.2 hold. Let $\bar{A}$ be the restriction of A to D_a and $\tilde{A}$ a maximal extension of $\bar{A}$ satisfying (4.1), which exists by the Zorn's lemma if we allow multivalued operators.

Lemma 4.3. *If* $A(u) \cap B_a \neq \emptyset$, *then* $u \in D_a$ *and* $\tilde{A}(u)$ *is single-valued, so that* $\tilde{A}(u) = \bar{A}(u)$.

According to the theory of the abstract non-linear contraction semigroups, the evolution equation

$$\begin{cases} du/dt \in A(u) \\ u(0) = u_0 \in D(A) \end{cases} \tag{4.2}$$

in Hilbert space $L^2(\Omega)$ admits a unique solution $u(t) \in D(A)$ satisfying (i) and (ii) in Theorem 3. Moreover it satisfies

$$\|D_t^+ u(t)\| \leqq \|D_t^+ u(0)\|. \tag{4.3}$$

Now take $u_0(x)$ such as $u_0(x) \in D_a$ and $\|A(u_0)\| < a$, then in virtue of

Lemma 4.3, we have $D_t^+u(0)=\tilde{A}(u_0)=\bar{A}(u_0)$ and $\|D_t^+u(t)\|<a$ by (4.3). Thus again in virtue of Lemma 4.3, we can see $D_t^+u(t)=\tilde{A}(u)=\bar{A}(u)$, which yields our result.

5. Supplementaries. Here we make mention of some generalization of our results, which can be carried out without any essential change of arguments stated above. Let $q(x, y, z_1, \cdots, z_n)$ be a real valued function satisfying

$$|q(x, y, z)| \leqq k_1(g(|y|)+|z|+|z|^{3-\delta}), \tag{5.1}$$

$$|q(x, y, z)-q(x, \eta, \zeta)| \leqq k_2\{h(|y|, |\eta|)|y-\eta|+(|z|^{2-\delta}+|\zeta|^{2-\delta})|z-\zeta|\}, \tag{5.2}$$

where g and h are locally bounded functions and $0<\delta \leqq 2$.

Consider the equations

$$\begin{cases} \sum_{j,k=1}^{n} \dfrac{\partial}{\partial x_j}\left(a_{jk}(x, u)\dfrac{\partial u}{\partial x_k}\right)+q(x, u, \operatorname{grad} u)-\lambda u=f \\ u|_{\partial\Omega}=\varphi, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t}=\sum_{j,k=1}^{n} \dfrac{\partial}{\partial x_j}\left(a_{jk}(x, u)\dfrac{\partial u}{\partial x_k}\right)+q(x, u, \operatorname{grad} u) \\ u(t, x)|_{\partial\Omega}=0, \quad u(0, x)=u_0(x). \end{cases} \tag{5.4}$$

Corresponding to Theorem $m(m=1, 2, 3)$, we have

Theorem 1′. *For any $a>0$, there exists a real number $\lambda_0=\lambda_0(a, k_1, k_2)$ such that the equation (5.3) has a solution $u(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ for any $\lambda \geqslant \lambda_0$ and for $f \in L^2(\Omega)$ and $\varphi \in H^{3/2}(\partial\Omega)$ with $\|f\|+|\varphi|_{3/2} \leqq a$.*

Theorem 2′. *In Theorem 1′, we can choose $\lambda_0=0$, if a, k_1, and k_2 are all sufficiently small.*

Theorem 3′. *If $\|u_0\|_2$ is sufficiently small and if k_1 and k_2 are also sufficiently small, then (5.4) has a unique solution.*

The more delailed exposition will be published elsewhere.

References

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