12. *The Stable Jet Range of Differential Complexes*

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1. Let M be an n-dimensional smooth manifold with countable basis. A topological space W is called an inductive vector bundle over M if there is an increasing sequence of finite-dimensional smooth vector bundles W_k $(k=0,1,\cdots)$ over M, W_k being a subbundle of W_{k+1}, such that $\lim \dim W_k=\infty$ and $W=\varinjlim W_k$ (inductive limit space). Then W becomes a fibre space over M. We can naturally define the space of smooth cross-sections $\Gamma(W)$ which has a module structure over the algebra $\mathcal{E}$ of smooth functions on M. We endow $\Gamma(W)$ with a nuclear topology such that, if M is compact, $\Gamma(W)$ coincides with the inductive limit space $\varinjlim \Gamma(W_k)$ where each $\Gamma(W_k)$ is assumed to have the C^∞-topology. Two inductive vector bundles W and W' are called isomorphic if $\Gamma(W)\cong\Gamma(W')$ as topological vector spaces and $\mathcal{E}$-modules.

We say that a sequence

$$0 \longrightarrow \textstyle\sum^0 \stackrel{d}{\longrightarrow} \sum^1 \stackrel{d}{\longrightarrow} \sum^2 \stackrel{d}{\longrightarrow} \cdots$$

is a differential complex over M if i) each $\sum^p$ is an $\mathcal{E}$-submodule of some $\Gamma(W^p)$, ii) d is continuous and $d\circ d=0$, iii) $\operatorname{supp} dL\subset \operatorname{supp} L$ where $\operatorname{supp} L$ means the support of $L\in\sum^p$.

2. Suppose that finite-dimensional smooth vector bundles E and F over M be given. Note that the jet bundles $J^k(E)$ of E $(k=0,1,2,\cdots)$ have the canonical surjective maps $\lambda^k: J^{k+1}(E)\to J^k(E)$. Hence we obtain the injective maps

$$(\lambda^k)^*: \operatorname{Hom}(J^k(E),F)\to \operatorname{Hom}(J^{k+1}(E),F)$$

$(k=0,1,2,\cdots)$, and thus the inductive vector bundle

$$C^1(E,F)=\varinjlim \operatorname{Hom}(J^k(E),F)$$

is constructed. The cross-section space of $C^1(E,F)$ is regarded as the space of the differential operators from $\Gamma(E)$ to $\Gamma(F)$.

More generally, set

$$C^p(E,F)=\varinjlim \operatorname{Hom}(\wedge^p J^k(E),F), \qquad p=1,2,\cdots$$
$$C^0(E,F)=F,$$

and write $C^p[E,F]=\Gamma(C^p(E,F))$ for $p=0,1,\cdots$.

Proposition. *Each $C^p[E,F]$ is canonically identified with the space of continuous multilinear alternating mappings from $\Gamma(E)\times\cdots\times\Gamma(E)$ (p times) to $\Gamma(F)$ satisfying the condition*

$$\operatorname{supp} L(\xi_1,\cdots,\xi_p)\subset \operatorname{supp}\xi_1\cap\cdots\cap\operatorname{supp}\xi_p.$$

3. Our main concern is to study the cohomological structure of a differential complex with the form

$$(1)\qquad \cdots \xrightarrow{d} C^p[E,F] \xrightarrow{d} C^{p+1}[E,F] \xrightarrow{d} \cdots.$$

Putting

$$^{(k)}C^p[E,F] = \Gamma(\mathrm{Hom}\,(\wedge^p J^k(E), F)),$$

we say that the subcomplex with order k is well-defined if $d(^{(k)}C^p[E,F]) \subset {}^{(k)}C^{p+1}[E,F]$ for $p=0,1,\cdots$ and thus the subcomplex

$$(2)\qquad \cdots \xrightarrow{d} {}^{(k)}C^p[E,F] \xrightarrow{d} {}^{(k)}C^{p+1}[E,F] \xrightarrow{d} \cdots$$

is meaningful. We denote by $H^*(E,F)=\sum\oplus H^p(E,F)$ and $^{(k)}H^*(E,F) = \sum\oplus {}^{(k)}H^p(E,F)$ the cohomology group of (1) and (2) respectively.

Definition 1. The complex (1) has the stable jet range $k \geqq k_0$ if, for $l \geqq l_0$,

i) the subcomplexes with order k are all well-defined;

ii) the injective maps induce the isomorphisms

$$^{(k)}H^*(E,F) \cong H^*(E,F).$$

Definition 2. The complex (1) has the elliptic jet range $l \geqq l_0$ if, for $l \geqq l_0$,

i) the subcomplexes with order l are all well-defined;

ii) each subcomplex with order l gives an elliptic complex over M.

4. To obtain a complex with the form (1), we shall introduce the following notion:

Definition 3. $\Gamma(E)$ is called a Lie algebra over M, if there is a $\Phi \in C^2[E,F]$ such that $[\xi,\eta]=\Phi(\xi,\eta)$ satisfies the Jacobi identity (so that $\Gamma(E)$ becomes a Lie algebra).

Assume that $\Gamma(E)$ is a Lie algebra over M. If there is a representation φ (as Lie algebra) of $\Gamma(E)$ to $\mathrm{Hom}\,(\Gamma(F),\Gamma(F))$ with $\varphi \in C^1[E, C^1(F,F)]$, then, by virtue of the cohomology theory of Lie algebra, we can canonically obtain a differential complex with the form (1): that is, $d\colon C^p[E,F] \to C^{p+1}[E,F]$ is given by the following formula:

$$dL(\xi_1,\cdots,\xi_{p+1}) = \sum(-1)^{i-1}\varphi(\xi_i)L(\xi_1,\cdots,\check{\xi}_i,\cdots,\xi_{p+1}) + \sum_{i<j}(-1)^{i+j}L([\xi_i,\xi_j],\xi_1,\cdots,\check{\xi}_i,\cdots,\check{\xi}_j,\cdots,\xi_{p+1})$$

$(\xi_1,\cdots,\xi_{p+1} \in \Gamma(E))$; here use is made of the identification of $C^p[E,F]$ mentioned in Proposition.

5. Let $\tau(M)$ be the tangent bundle over M. Then $A(M)=\Gamma(\tau(M))$ (=the space of vector fields) becomes a Lie algebra over M under the natural bracket operation and admits $\mathcal{E}$ as the representation space via the usual differentiation. As M. V. Losik [2] has shown that the cohomology group of the differential complex induced from this representation is isomorphic to $H^*(B(\tau^C),\boldsymbol{R})$; here $B(\tau^C)$ denotes the principal $U(n)$-bundle over M associated to $\tau(M)\otimes\boldsymbol{C}$. Moreover, this complex has the stable jet range $\geqq 1$ and the elliptic jet range $\geqq 0$. (The subcomplex with order 0 is nothing but the de rham complex over M.)

We denote by $D(k)$ the space of the k-th differential operators on M and by $T(a, b)$ the tensor space with type (a, b) on M.

Theorem 1. *The cohomology groups of the differential complexes induced from the representations of $A(M)$ on $D(k)$ and $T(a, b)$ are described as follows:*

Representation space	$D(k)$	$T(a,b)$
Representation	Bracket as differential operator	Lie differentiation
Stable jet range	$\geqq k$	$\geqq \mathrm{Max}\{n(b-a+1),1\}$
Elliptic jet range	$\geqq k$	$\geqq 1$
Cohomology group	$H^*(B(\tau C),\boldsymbol{R})$	0 if $a>b$? otherwise

Let $\mathcal{E}^h$ for the h-dimensional trivial bundle over M.

$\Gamma(\tau(M)\oplus\varepsilon^h)$ admits a structure of Lie algebra over M, given by the bracket operation

$$\left[\xi\oplus\sum_{i=1}^h f_i, \eta\oplus\sum_{i=1}^h g_i\right]=[\xi, \eta]\oplus\sum_{i=1}^h (\xi g_i-\eta f_i).$$

This Lie algebra, denoted by $D_h(1)$, operates on $\mathcal{E}$ in two ways such that

(3) $$(\xi\oplus\sum f_i)\varphi=\xi\varphi$$

(4) $$(\xi\oplus\sum f_i)\varphi=\xi\varphi+\sum f_i\varphi,$$

each of which gives a representation of $D_h(1)$ on $\mathcal{E}$. Corresponding to these representations, we obtain the two differential complexes:

(3′) $$\cdots\longrightarrow C^p[\tau(M)\oplus\varepsilon^h, \varepsilon^1]\xrightarrow{d'} C^{p+1}[\tau(M)\oplus\varepsilon^h, \varepsilon^1]\longrightarrow\cdots$$

(4′) $$\cdots\longrightarrow C^p[\tau(M)\oplus\varepsilon^h, \varepsilon^1]\xrightarrow{d} C^{p+1}[\tau(M)\oplus\varepsilon^h, \varepsilon^1]\longrightarrow\cdots.$$

Theorem 2. i) *The differential complex* (3′) *has the stable jet range* $\geqq 1$ *and the elliptic jet range* $\geqq 0$; *its cohomology group is isomorphic to $H^*(B(\tau^C)\times T^h, \boldsymbol{R})$ where T^h denotes the h-dimensional torus.*

ii) *The differential complex* (4′) *has the stable jet range* $\geqq 1$ *and the elliptic jet range* $\geqq 0$.

The details will be discussed in the forthcoming paper.

References

[1] I. M. Gelfand and D. B. Fuks: Cohomologies of Lie algebra of tangential vector fields on a smooth manifold. I, II. Functional Analysis and their Application, **3**, 194–210 (1969); **4**, 110–116 (1970).

[2] M. V. Losik: On the cohomologies of infinite-dimensional Lie algebras of vector fields. Functional Analysis and their Application, **4**, 127–135 (1970).