220. A Remark on the Spectral Order of Operators

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1. Spectral order. Consider a Hilbert space $\mathfrak{H}$. Let $\mathcal{S}$ be the set of all hermitean (bounded linear) operators acting on $\mathfrak{H}$. Suppose that $A, B \in \mathcal{S}$. Then the spectral theorem assures us that there are the spectral resolutions of the identity $E(t)$ and $F(t)$ such as

$$(1) \qquad A=\int_{-\infty}^{\infty} tdE(t) \quad \text{and} \quad B=\int_{-\infty}^{\infty} tdF(t).$$

Very recently, besides the usual order of hermitean operators, M. P. Olson [4] introduce a new order in the following manner:

Definition A. If A and B are hermitean operators as in (1), then $A \preceq B$ provided that

$$(2) \qquad E(t) \geqq F(t) \qquad \text{for } -\infty < t < \infty.$$

This new order is called the *spectral order* by Olson. He proves, among others, that the spectral order differs from the usual one (coincides if A commutes with B) and $\mathcal{S}$ becomes a conditionally complete lattice under the spectral order.

In the present note, we shall introduce an another definition of the spectral order (cf. [4, Cor. 1]):

Definition B. If A and B are hermitean operators, then $A \ll B$ if and only if

$$(3) \qquad f(A) \leqq f(B)$$

for any continuous monotone (nondecreasing) function f defined on an interval which contains $\sigma(A) \cup \sigma(B)$, where $\sigma(C)$ is the spectrum of C.

In the below, we shall reconstruct Olson's theory under Definition B.

2. Properties. The following theorem is our main result:

Theorem 1. *Definitions A and B are equivalent.*

Proof. Suppose that $A \preceq B$ in the sense of Olson. Suppose furthermore that A and B are expressed in (1). Then $E(t) \geqq F(t)$ for every t. If f is a continuous monotone function, then we have

$$\begin{aligned}(f(A)x \mid x) &= \int_a^b f(t)d(E(t)x \mid x) \\ &= f(b)\|x\|^2 - \int_a^b \|E(t)x\|^2 \, df(t) \\ &\leqq f(b)\|x\|^2 - \int_a^b \|F(t)x\|^2 \, df(t)\end{aligned}$$

$$=\int_a^b f(t)d(F(t)x\,|\,x)$$
$$=(f(B)x\,|\,x)$$

for every $x\in\mathfrak{H}$. Hence we have $f(A)\leqq f(B)$ so that $A\ll B$ by Definition B.

Conversely, let us suppose that $A\ll B$. Define

$$f_n(s)=\begin{cases}0 & \text{for } s\leqq t\\ 1 & \text{for } s\geqq t+1/n\\ \text{linear between} & (t, t+1/n).\end{cases}$$

Then we have

$$f_n(A)\to 1-E(t) \quad \text{strongly,}$$
$$f_n(B)\to 1-F(t) \quad \text{strongly,}$$

and

$$f_n(A)\leqq f_n(B) \qquad \text{for } n=1, 2, \cdots.$$

Hence we have $E(t)\geqq F(t)$ for every t or$A\preceq B$.

In the remainder of this section, we shall derive some properties of the spectral order from Definition B.

Theorem 2. *If $A\ll B$ then $A\leqq B$.*

Proof. Put $f(t)=t$. Then Definition B implies at once $A\leqq B$.

Remark. The converse is not true in general. If $f(A)\leqq f(B)$ whenever $A\leqq B$, then f is called a *monotone operator function* in the sense of Loewner, cf. [1]. There is a monotone function which is not operator monotone, so that the converse of Theorem 1 is not valid.

Theorem 3. *If A and B are mutually commutative hermitean operators, then $A\leqq B$ implies $A\ll B$.*

This is a consequence of Gelfand's theory of representation of commutative Banach algebras.

Remark. The implication that Definition A implies Theorem 3 is known, cf. [5, Ex. 3–B, No. 9].

3. Lattice completeness. A novelty of Olson's theory is that the space $\mathcal{S}$ forms a conditionally complete lattice. However, this is a consequence of the following

Theorem 4 (Iwamura [3, p. 25]). *If L is a lattice with 0 and 1, and if L' is a complete lattice, then the set J of all join-homomorphisms of L into L' is a complete lattice under the natural ordering:*

(4) $\quad f\leqq g$ *if and only if* $f(x)\leqq g(x)$ *for all* $x\in L$.

Original paper of Iwamura (in Japanese) is published in 1942. Afterwards, Grätzer and Schmidt [2] obtained the theorem for join-endomorphisms. Their proofs are the same.

Proof. Put

$$g(x)=\vee_\alpha f_\alpha(x)$$

where $\{f_\alpha\}$ is the given set of join-homomorphisms. Then

$$g(x)\cup g(y)=\vee_\alpha f_\alpha(x)\cup\vee_\alpha f_\alpha(y)=\vee_\alpha(f_\alpha(x)\cup f_\alpha(y))$$
$$=\vee_\alpha f_\alpha(x\cup y)=g(x\cup y),$$

so that g is a join-homomorphism. Cleary we have $f_\alpha\leqq g$ for all α. Moreover, if $h\in J$ satisfies $h\geqq f_\alpha$, then $h(x)\geqq f_\alpha(x)$ for all x. Hence we have $h\geqq g$ or $g=\vee_\alpha f_\alpha$. Since J contains clearly 0 and 1, we can conclude that J is complete.

References

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