156. *On K-Souslin Spaces*

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A. Martineau defined in [1] the K-Souslin spaces as a generalization of the Souslin spaces. In this paper we shall show that the K-Souslin spaces coincide with the quasi-Souslin spaces defined in [2].

Let E be a topological space, $\mathcal{P}(E)$ the set of all subsets of E, and $\mathcal{K}(E)$ the set of all non-empty compact subsets of E. We consider $\mathcal{P}(E)$ as the topological space where $\mathcal{P}(U)$ for all open sets U of E constitutes a basis of the open sets for $\mathcal{P}(E)$, and we consider in $\mathcal{K}(E)$ the relative topology of that of $\mathcal{P}(E)$.

A Hausdorff topological space E is said to be a *K-Souslin space* if there exist a complete separable metric space P and a continuous mapping φ from P to $\mathcal{K}(E)$ such that $E=\bigcup_{p\in P}\varphi(p)$.

Proposition 1. *Every quasi-Souslin space E is a K-Souslin space.*

Proof. Since E is a quasi-Souslin space, there exists a defining S-filters $\Phi_m(m=1,2,\cdots)$ such that each Φ_m has a filter base

$$S_n^{(m)}(n=1,2,\cdots).$$

For any sequence $n_i(i=1,2,\cdots)$ of natural numbers, $E_{n_1,n_2,\cdots,n_i}=(S_{n_1}^{(1)})^c\cap(S_{n_2}^{(2)})^c\cap\cdots\cap(S_{n_i}^{(i)})^c$ converges for $i\to\infty$ to the compact set $\bigcap_i E^-_{n_1,n_2,\cdots,n_i}$ in $\mathcal{P}(E)$, since every ultrafilter containing all $E_{n_1,n_2,\cdots,n_i}$ converges.

Let P be the set of all sequences of natural numbers, that is $P=\prod_{i=1}^{\infty}N_i$ where each $N_i=N$, the set of all natural numbers with the discrete topology. Then P is a complete separable metric space.

Now we define a mapping φ from P to $\mathcal{K}(E)$ by $\varphi(p)=\bigcap_i E^-_{n_1,n_2,\cdots,n_i}$ for all $\{n_i\}=p\in P$. Then we can see easily that φ is continuous and $E=\bigcup_{p\in P}\varphi(p)$.

Proposition 2. *Every K-Souslin space is a quasi-Souslin space.*

Proof. It is sufficient to prove for any Hausdorff topological space E the following fact.

If φ be a continuous mapping from a quasi-Souslin space F to $\mathcal{K}(E)$ and $E=\bigcup_{x\in F}\varphi(x)$, then E is a quasi-Souslin space.

Then, it is sufficient to prove that the subset

$$D=\{(x,y)\,|\,x\in F,\,y\in\varphi(x)\}$$

of $F\times E$ is quasi-Souslin, because E is the image of D by the projection from $F\times E$ to E.

Let q be the restriction to D of the projection from $F\times E$ to F, then $q(D)=F$.
Let $\Phi_n(n=1,2,\cdots)$ be a defining S-filters in F, then $q^{-1}(\Phi_n)$ are S-filters of D.
Let Ψ be any ultrafilter of D disjoint from every $q^{-1}(\Phi_n)$, then for each n there exists an A_n in Φ_n such that $q^{-1}(A_n)\cap B_n=\phi$ for some B_n in Ψ, therefore $A_n\cap q(B_n)=\phi$. That is, for the ultrafilter $q(\Psi)$ of F, we have $q(\Psi)\not\supset\Phi_n$ for each n. Therefore $q(\Psi)$ converges to some element x in F.
Then to prove that Ψ is convergent in D, it is sufficient to show that any neighbourhood W of the compact set $\{x\}\times\varphi(x)$ is a member of Ψ.
Now, since φ is continuous mapping from F to $\mathscr{P}(E)$, we can find a neighbourhood of U of x and an open set V including $\varphi(x)$ such that $W\supset(U\times V)\cap D$ and $V\supset\varphi(U)$. Since $q(\Psi)$ converges to x, there exists a member B in Ψ such that $q(B)\subset U$.
Then, from $q(U)\subset V$, we can conclude

$$B\subset(U\times\varphi(U))\cap D\subset(U\times V)\cap D\subset W.$$

Thus the proof is completed.

References

[1] A. Martineau: Sur des théorémes de S. Banach et L. Schwartz concernant le graphe fermé. Studia Mathematica, **30**(1), 43–51 (1968).

[2] M. Nakamura: On quasi-Souslin space and closed graph theorem. Proc. Japan Acad., **46**(6), 514–517 (1970).