121. On the Global Existence of Real Analytic Solutions of Linear Differential Equations. I

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In this note we discuss the problem of global existence of real analytic solutions of linear differential equations $P(D)u=f$ *with constant coefficients*. This problem has remained unsolved by the reason that the topology of the space of real analytic functions on an open set Ω in $\boldsymbol{R}^n$ is complicated (Ehrenpreis [2], Martineau [8]). In fact there has been no general result even when Ω is convex; the only results hitherto known seem to be Theorems α and β below.

The recent theory of sheaf $\mathcal{C}$ (Kashiwara [4], Sato [9]-[11]) however provides us an effective tool on this subject considered thus far to be very difficult, as will be described in this note. The details and complete arguments will be given somewhere else.

Throughout this note we denote by Ω an open set in $\boldsymbol{R}^n$ and by $\bar{\Omega}$ and $\partial\Omega$ its closure and boundary, respectively. $\mathcal{A}$ and $\mathcal{B}$ denote the sheaf of germs of real analytic functions and that of hyperfunctions, respectively. We denote by $\mathcal{A}(\Omega)$ the space of real analytic functions on Ω. For a compact set K in $\boldsymbol{R}^n$ we also denote by $\mathcal{A}(K)$ the space of real analytic functions on K, i.e., $\mathcal{A}(K)=\varinjlim_{V\supset K}\mathcal{O}(V)$, where V is a complex neighbourhood of K and $\mathcal{O}(V)$ denotes the space of holomorphic functions on V. Since we need not know the topological structure of these spaces in our arguments, we do not discuss it here.

We first list up two known theorems for the reader's convenience.

Theorem α (Malgrange [7]). *Let $P(D)$ be elliptic, i.e., have a principal symbol $P_m(\xi)$ never vanishing for any non-zero real cotangent vector ξ. Then for any open set Ω in $\boldsymbol{R}^n$ $P(D)u=f$ has a solution $u(x)$ in $\mathcal{A}(\Omega)$ for any $f(x)$ in $\mathcal{A}(\Omega)$.*

Theorem β (Ehrenpreis). *Let K be a compact convex set in $\boldsymbol{R}^n$. Then $P(D)u=f$ has a solution $u(x)$ in $\mathcal{A}(K)$ for any $f(x)$ in $\mathcal{A}(K)$.*

In the rest of this note we always assume that Ω is a relatively compact domain in $\boldsymbol{R}^n$.

Theorem 1. *Assume that Ω is represented as $\{x\,|\,\varphi(x)<0\}$ by a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\bar{\Omega}$ satisfying* $\mathrm{grad}_x\,\varphi\neq 0$ *on $\partial\Omega$. Assume further that $P(D)$ satisfies the following condition* (1) *and Ω satisfies condition* (2). *Then for any*

$f\in\mathcal{A}(\bar{\Omega})$, we find a real analytic solution $u(x)$ of $P(D)u=f$ in Ω.

(1) *The principal symbol $P_m(\xi)$ of $P(D)$ is real and of simple characteristics, i.e., $\mathrm{grad}_\xi P_m\neq 0$ whenever $P_m(\xi)=0$, where ξ is a non-zero real cotangent vector.*

(2) *At any $x_0\in\partial\Omega$ the bicharacteristic curve $b_{(x_0,\mathrm{grad}_x\varphi)}$ of $P(D)$ issuing from $(x_0, \mathrm{grad}_x\varphi|_{x_0})$ intersects Ω in an open interval.*

Note that if $P_m(\mathrm{grad}_x\varphi|_{x_0})\neq 0$ condition (2) is vacuous.

This theorem is proved using the elementary solution constructed in Kawai [5] and the theory of sheaf $\mathcal{C}$ (Sato [11]). We remark that the flabbiness of sheaf $\mathcal{C}$ (Kashiwara [4]) is effectively used in the proof. Note that Andersson [1] has also constructed a "good" elementary solution for some class of linear differential operators with constant coefficients, which he calls locally hyperbolic operators. We remark that the conditions on lower order terms assumed in Andersson [1] can be removed using the theory of hyperfunctions.

Theorem 1′. *Assume that a compact set K has a fundamental system of neighbourhoods with smooth boundary Ω_ν, where Ω_ν is represented as $\{x\,|\,\varphi_\nu(x)<0\}$ by a real valued real analytic function $\varphi_\nu(x)$ defined in a neighbourhood of $\bar{\Omega}_\nu$ and satisfies following condition:*

(2′) *At any boundary point x_0 of Ω_ν the bicharacteristic curve of $P(D)$ issuing from $(x_0, \mathrm{grad}_x\varphi_\nu|_{x_0})$ intersects K in a closed interval.*

Then $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ holds.

We next consider the case where the principal symbol $P_m(\xi)$ is not real.

We denote by $A_m(\xi)$ and $B_m(\xi)$ the real part and the imaginary part of $P_m(\xi)$, respectively. We assume in Theorem 2 that

(3) $\mathrm{grad}_\xi A_m$ and $\mathrm{grad}_\xi B_m$ are linearly independent on $\{\xi\in\boldsymbol{R}^n-\{0\}\,|\,P_m(\xi)=0\}$.

A bicharacteristic plane through (x_0, ξ^0) is by definition the 2-dimensional linear variety passing through x_0 which is spanned by $\mathrm{grad}_\xi A_m|_{\xi=\xi^0}$ and $\mathrm{grad}_\xi B_m|_{\xi=\xi^0}$ where ξ^0 is a non-zero real cotangent vector satisfying $P_m(\xi^0)=0$.

Theorem 2. *Assume that $\Omega=\{x\,|\,\varphi(x)<0\}$, where $\varphi(x)$ is a real valued real analytic function defined near $\bar{\Omega}$ such that $\mathrm{grad}_x\varphi(x)\neq 0$ on $\partial\Omega$, satisfies the following condition (4). Then $P(D)u=f$ has a solution $u(x)$ in $\mathcal{A}(\Omega)$ for any $f(x)$ in $\mathcal{A}(\bar{\Omega})$.*

(4) *At any boundary point x_0 of Ω with $P_m(\mathrm{grad}_x\varphi|_{x_0})=0$ the bicharacteristic plane passing through $(x_0, \mathrm{grad}_x\varphi|_{x_0})$ never intersects Ω.*

This is proved using a result of Kawai [6]. We can also restate this theorem in a form analogous to Theorem 1′ so that it assures the existence of solutions in the space of $\mathcal{A}(K)$ for a compact set K. We omit the details here.

Remark. When K is a compact set in $\boldsymbol{R}^2$, we can prove that $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ holds under the following condition (5), without assuming any conditions on $P(D)$.

(5) Any characteristic line of $P(D)$ intersects K in a closed interval.

Now we consider the case where $f(x)$ belongs to $\mathcal{A}(\Omega)$, not necessarily to $\mathcal{A}(\bar{\Omega})$.

Theorem 3. *Assume that the space dimension n is equal to* 2. *Let a domain Ω satisfy the following condition*:

(6) *Any characteristic line of $P(D)$ intersects Ω in an open interval. Then $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds.*

Remark. In the above theorem we need not pose any conditions on $P(D)$ and the regularity of the boundary of Ω. This condition (6) was found by Hörmander [3] (and Ehrenpreis) in the discussion of global existence of C^∞-solutions. Note that the notion of characteristic line coincides with that of bicharacteristic curve when $n=2$.

Now using the notion of bicharacteristic curve we have the following Theorem 4 in the higher dimensional case. (Cf. Hörmander [3] Theorem 3.7.6., where the global existence of distribution solutions is treated.)

Theorem 4. *Assume that Ω is represented as $\{x \mid \varphi(x)<0\}$ by a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\bar{\Omega}$ satisfying* $\operatorname{grad}_x \varphi(x)\neq 0$ *on $\partial\Omega$. Assume further that $P(D)$ satisfies condition* (1) *and that Ω satisfies condition* (2) *and condition* (7), *which is given in the below. Then $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds.*

(7) *There exists a family of open sets $\{N_j\}_{j=1}^p$ which satisfies the following*:

For any point x in $\partial\Omega$ we can find some j such that for any bicharacteristic curve $b_{(x,\xi)}$ of $P(D)$ through (x,ξ) $b_{(x,\xi)}\cap(\bar{\Omega}-\{x\})\cap N_j$ is connected, where N_j is a neighbourhood of x and ξ is a non-zero real cotangent vector.

This theorem is proved by using the elementary solution constructed in Kawai [5] and the theory of sheaf $\mathcal{C}$ (Sato [11]). We remark that the flabbiness of sheaf $\mathcal{B}$ and also that of sheaf $\mathcal{C}$ (Kashiwara [4]) are effectively used in the proof.

Remark 1. In condition (7) we have assumed the local strict convexity of Ω on half of the bicharacteric curve which touches Ω, so to speak. We can also prove the theorem under the following condition (8) instead of condition (7). Condition (8) requires, so to speak, the global (but not necessarily strict) convexity of Ω on half of bicharacteristic curve.

(8) For any bicharacteristic curve $b_{(x,\xi)}$ of $P(D)$ through (x, ξ) intersects Ω in an open interval, where x belongs to $\partial\Omega$ and ξ is a non-zero real cotangent vector.

Remark 2. We need not necessarily assume the real analyticity of $\varphi(x)$ in condition (7) if we assume condition (2′) (, where we take $\bar{\Omega}$ as K) instead of condition (2). In fact it is sufficient to assume in condition (7) that $\varphi(x)$ is a C^1-function.

The extension of results of this note to the case where $P(D)$ is not of simple characteristics, to the case where P is a linear differential operator with real analytic coefficients and to the case where P is an overdetermined system will be given in our forthcoming notes.

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