122. *Hyperfunction Solutions of the Abstract Cauchy Problem*

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In this note we will report a few results on hyperfunction solutions of the abstract Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) \\ u(0) = a \end{cases}$$

where A is a closed linear operator in a Banach space X and $a \in X$. We discuss conditions for existence, uniqueness and regularity.

Hyperfunctions defined as boundary values of holomorphic functions are more general than Schwartz distributions. Hence if the Cauchy problem is well-posed in the sense of distribution, then we shall find that it is also well-posed in the sense of hyperfunction.

Distribution solutions were investigated by J. Chazarain [1], G. Da Prato, U. Mosco [2], D. Fujiwara [3], J. L. Lions [4] and T. Ushijima [7].

A complete proof will be published elsewhere.

§1. Hyperfunctions with values in a Banach space. We shall use vector valued hyperfunctions of one variable. The case of one variable is much simpler than that of many variables. We refer the reader to M. Sato [6] for the scalar case.

Let E be a Banach space. Consider the space $\mathcal{O}(\Omega, E)$ of all E-valued holomorphic functions defined on Ω, where Ω is an open subset in $\boldsymbol{C}^1$. Let S be an open set in $\boldsymbol{R}^1$. We define an E-valued hyperfunction to be an element of the quotient space:

$$\mathcal{B}(S, F) = \frac{\mathcal{O}(D-S, E)}{\mathcal{O}(D, E)},$$

where D is a complex neighbourhood of S, which contains S as a closed set.

For E-valued hyperfunctions we can establish results similar to the case of scalar hyperfunctions. In particular $\mathcal{B}(S, E)$ does not depend on the complex neighbourhood D of S. The E-valued hyperfunctions are flabby, that is, for every $f \in \mathcal{B}(S, E)$ there exists $\mathcal{F} \in \mathcal{B}(R^1, E)$ such that the restriction $\mathcal{F}|_S$ of $\mathcal{F}$ to S coincides with f. The notion of support can be defined. These facts can be proved in a way analogous to M. Sato [6] with the aid of Runge's approximation theorem of E-

valued holomorphic functions.

§2. Well-posedness of the Cauchy problem. Let E and F be Banach spaces whose norms are $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively. $L(E, F)$ is a Banach space consisting of all bounded linear operators from E to F equipped with the operator norm denoted by $\|\cdot\|_{E\to F}$. The set $L(E, E)$ is written $L(E)$ for short.

Let X be a Banach space, and A a closed linear operator in X. The definition domain of A with the graph norm is a Banach space and is denoted by $[D(A)]$. $\rho(A)$ means the resolvent set of A. In the following I is the identity mapping, in particular we shall use notations I_X and $I_{[D(A)]}$ which are the identity on X and on $[D(A)]$ respectively.

Definition. A closed operator A is said to be well-posed for the Cauchy problem at $t=0$ in the sense of hyperfunction (well-posed, for short), if there exists $T \in \mathscr{B}(R^1, L(X, [D(A)]))$ satisfying the following conditions:

(2.1) support of $T \subset [0, \infty)$;

(2.2) $(\delta^{(1)}(t)\otimes I - \delta(t)\otimes A)*T = \delta(t)\otimes I_X$,
$T*(\delta^{(1)}(t)\otimes I - \delta(t)\otimes A) = \delta(t)\otimes I_{[D(A)]}$

where $\delta(t-\tau)$ is the Dirac measure at $t=\tau$, $*$ means convolution, $\delta^{(k)}(t)$ k-th derivative of $\delta(t)$ and $\otimes$ tensor product.

We shall call T in the definition a fundamental solution.

Remark 1. If A is well-posed, then the fundamental solution T is unique in $\mathscr{B}(R^1, L(X, [D(A)]))$. This result easily follows from the facts that T is a two-sided fundamental solution and its support is contained in $[0, \infty)$.

Our first theorem is the following.

Theorem 1. *A closed linear operator A is well-posed if and only if the resolvent of A satisfies the condition*:

(2.3) *For any $\varepsilon>0$ there exists K_ε such that $\sum_\varepsilon=\{\lambda\,;\,\mathrm{Re}\,\lambda \geq \varepsilon|\mathrm{Im}\,\lambda| + K_\varepsilon\}$ is contained in $\rho(A)$ and in this set $\|(\lambda-A)^{-1}\|_{X\to X} \leq C_\varepsilon \exp(\varepsilon|\lambda|)$ holds.*

Outline of proof of Theorem 1.

Necessity. Let A be well-posed. Then, by the definition, we have a hyperfunction fundamental solution T. Take a hyperfunction $T_1 \in \mathscr{B}(R^1, L(X, [D(A)]))$ such that

$$\begin{aligned} t<1 \qquad & T_1 = T, \\ t>1 \qquad & T_1 = 0. \end{aligned}$$

Such a hyperfunction T_1 exists, because of the flabbiness of the hyperfunctions. By (2.2) and the property of T_1 we have

$$(\delta^{(1)}(t)\otimes I - \delta(t)\otimes A) * T_1 = \delta(t)\otimes I_X + \sum_{n=0}^{\infty} \delta^{(n)}(t-1)\otimes A_n$$

where $A_n \in L(X)$, and according to the theory of hyperfunctions, for every $\varepsilon>0$ there exists M_ε such that $\left\|\sum_{n=0}^{\infty}\lambda^n A_n\right\|_{X\to X}\leq M_\varepsilon \exp(\varepsilon|\lambda|)$. Thus Laplace transform of T_1, $\langle T_1, \exp(-\lambda t)x\rangle$, $x\in X$, satisfies

$$(\lambda-A)\langle T_1, \exp(-\lambda t)x\rangle = x+\sum_{n=0}^{\infty}\lambda^n \exp(-\lambda)A_n x.$$

If $\operatorname{Re}\lambda\geq\varepsilon|\lambda|+\log 2M_\varepsilon$, then $\left\|\sum_{n=0}^{\infty}\lambda^n \exp(-\lambda)A_n\right\|_{X\to X}\leq\frac{1}{2}$. Hence the resolvent of A exists and on that domain

$$\|(\lambda-A)^{-1}\|_{X\to[D(A)]}\leq 2\,\|\langle T_1, \exp(-\lambda t)\cdot\rangle\|_{X\to[D(A)]}\leq C_\varepsilon \exp(2\varepsilon|\lambda|).$$

Sufficiency. Fix a real $\omega\in\rho(A)$.

Define paths $\Gamma_{\varepsilon+}$: if $\omega\leq\operatorname{Re}\lambda\leq K_\varepsilon$, $\operatorname{Im}\lambda=0$.
if $\operatorname{Re}\lambda\geq K_\varepsilon$, $\operatorname{Re}\lambda=\varepsilon\operatorname{Im}\lambda+K_\varepsilon$ and $\operatorname{Im}\lambda\geq 0$
$\Gamma_{\varepsilon-}$: if $\omega\leq\operatorname{Re}\lambda\leq K_\varepsilon$, $\operatorname{Im}\lambda=0$.
if $\operatorname{Re}\lambda\geq K_\varepsilon$, $\operatorname{Re}\lambda=-\varepsilon\operatorname{Im}\lambda+K_\varepsilon$ and $\operatorname{Im}\lambda\leq 0$.

Put $T_+(z)=\frac{1}{2\pi i}\int_{\Gamma_{\varepsilon+}} e^{\lambda z}(\lambda-A)^{-1}d\lambda$ and $T_-(z)$ similarly. By varying paths $\Gamma_{\varepsilon\pm}(\varepsilon>0)$, one holomorphic function $T(z)$ is defined on $C^1-[0,\infty)$ which is an analytic continuation of $T_\pm(z)$. More precisely it belongs to $\mathcal{O}(C^1-[0,\infty), L(X,[D(A)]))$. It is easy to check

$$\frac{dT(z)}{dz}=AT(z)+\frac{-1}{2\pi i}\frac{e^{\omega z}}{z}I_X \qquad \text{on } X,$$

$$\frac{dT(z)}{dz}=T(z)A+\frac{-1}{2\pi i}\frac{e^{\omega z}}{z}I_{[D(A)]} \qquad \text{on } [D(A)].$$

This shows that T is a desired hyperfunction fundamental solution.

Remark 2. J. Chazarain [1] characterized well-posedness of the abstract Cauchy problem in the sense of distribution. Comparing Theorem 1 with his result, we conclude that operators which are well-posed in the sense of hyperfunction contain those which are well-posed in the sense of distribution.

§3. Regularity. As for regularity of the hyperfunction fundamental solution, we have the following:

Theorem 2. *A closed operator A is well-posed and its fundamental solution is holomorphic in the sector $\Sigma=\left\{z\,;\,|\arg z|<\alpha, 0<\alpha<\frac{\pi}{2}\right\}$, if and only if A satisfies the following conditions:*

For any $\varepsilon>0$, there exists a real ω_ε, and for any λ in the sector $\Sigma_\varepsilon=\left\{\lambda\,;\,|\arg(\lambda-\omega_\varepsilon)|<\theta, \theta=\frac{\pi}{2}+\alpha-\varepsilon\right\}$, we have $(\lambda-A)^{-1}\in L(X)$ with the estimate $\|(\lambda-A)^{-1}\|_{X\to X}\leq C_\varepsilon\exp(\varepsilon|\lambda|)$.

Theorem 3. *A closed operator A is well-posed and its fundamental solution is real analytic on the positive real axis, if and only if A satisfies the condition:*

For any $\varepsilon>0$ *there exists* K_ε *and* $0<\delta_\varepsilon\leq\varepsilon$, *and for any* λ *in the set* $\Sigma_\varepsilon=\{\lambda\,;\,\varepsilon\,\mathrm{Re}\,\lambda\geq-\delta_\varepsilon|\mathrm{Im}\,\lambda|+K_\varepsilon\}$, $(\lambda-A)^{-1}\in L(X)$ *exists and the estimate* $\|(\lambda-A)^{-1}\|_{X\to X}\leq C_\varepsilon\exp(\varepsilon|\mathrm{Re}\,\lambda|+\delta_\varepsilon|\mathrm{Im}\,\lambda|)$ *holds.*

Remark 3. The criterion which correspond to Theorem 2 in the case of distributions solutions was given by G. Da Prato, U. Mosco [2] and D. Fujiwara [3].

On C_0-semigroups K. Masuda [5] obtained the result corresponding to Theorem 3.

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