36. *On Kodaira Dimensions of Certain Algebraic Varieties*

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1. In this note we give our results on *Kodaira dimensions* of subvarieties of abelian varieties and of varieties which are images of rational maps of abelian varieties. Details will be published elsewhere.

2. In this note all algebraic varieties are assumed to be irreducible and defined over $\boldsymbol{C}$.

Let V be a complete algebraic variety of dimension n, and let $\tilde{V}$ be a non-singular model of V. For any positive integer m, the *m-genus* $p_m(V)$ and the *irregularity* $q(V)$ of V are defined by

$$p_m(X)=\dim_C H^0(\tilde{V},(\Omega^n_{\tilde{V}})^{\otimes m}),$$
$$q(V)=\dim_C H^0(V,\Omega^1_{\tilde{V}}),$$

where $\Omega^k_{\tilde{V}}$ is the sheaf of germs of holomorphic k forms on $\tilde{V}$. Instead of $p_1(V)$, we use the notation $p_g(V)$ and call it the *geometric genus* of V. $p_m(V)$ and $q(V)$ are independent of the choice of the non-singular model $\tilde{V}$ of V.

S. Iitaka [1] proved that if there exists a positive integer m_0 such that the inequality $p_{m_0}(V)>1$ holds, then the inequality

$$\alpha m^{\kappa(V)} \leqq p_{mm_0}(V) \leqq \beta m^{\kappa(V)}$$

holds for every sufficiently large integer m, where α, β are positive numbers and $\kappa(V)$ is a positive integer uniquely determined by the plurigenera of V. When the inequality $p_m(V)\leqq 1$ holds for every positive integer m and the equality holds for at least one integer m, we define $\kappa(V)=0$. When all plurigenera vanish, we define $\kappa(V)=-\infty$.

As every plurigenus is a birational invariant of $V, \kappa(V)$ is also a birational invariant of V. $\kappa(V)$ is called the *Kodaira dimension* of the variety V.

3. Let B be a subvariety of an abelian variety A. Then we have $\kappa(B)\geqq 0$. This implies the well known fact that unirational and ruled varieties cannot be embedded into abelian varieties. For this subvariety B, we have the following theorem.

Theorem 1. *The following conditions are equivalent*:

1) $p_g(B)=1$

2) $\kappa(B)=0$

3) *The subvariety B is a translations of an abelian subvariety B' of A by an element $a\in A$.*

Corollary 1. *We assume that B goes through the origin of A.*

If B generates A, then we have

$$\kappa(B)>0.$$

Corollary 2. *If*

$$q(B)>\dim B,$$

then we have

$$\kappa(B)>0.$$

This theorem also implies the following propositions.

Proposition 1. *Let V be a complete algebraic variety. If $q(V)>\dim V$ and if there exists a regular point $x\in V$ at which holomorphic 1-forms generates $\Omega^1_{V,x}$ as $\mathcal{O}_{V,x}$ module, then we have*

$$\kappa(V)>0.$$

Proposition 2. *Let B be a subvariety of an abelian variety A. We assume that the Kodaira dimension $\kappa(B)$ of B is positive. Then the equality*

$$\kappa(B)=\dim B-l$$

holds if and only if B has an algebraic system of l-dimensional abelian varieties with index 1 and has no algebraic systems of abelian varieties of dimension $>l$.

Let V be a complete algebraic variety. We call V an algebraic variety of hyperbolic type (general type) if $\kappa(V)=\dim V$.

Corollary. *Let B be a subvariety of a simple abelian variety. Then B is of hyperbolic type.*

For a smooth subvariety B of an abelian variety every pluricanonical system $|mK_B|$ is free from fixed components and base points.

Theorem 2. *Let B be a smooth subvariety of hyperbolic type of an abelian variety. Then the canonical divisor of B is ample.*

4. We consider a complete algebraic variety which is an image of a rational map of an abelian variety.

Proposition 3. *Let V be a complete algebraic variety such that there exists a generically surjective rational map f of an abelian variety A to V. Then the Kodaira dimension $\kappa(V)$ of V is 0 or $-\infty$.*

Example. Let E_ρ be the elliptic curve with the fundamental periods 1 and $\rho=e^{2\pi\sqrt{-1}/3}$. The abelian variety $A=E_\rho\times E_\rho\times E_\rho$ has an analytic automorphism

$$g:(z_1,z_2,z_3)\rightarrow(\rho z_1,\rho z_2,\rho z_3).$$

The automorphism g generates a cyclic group G of order three and the quotient analytic space A/G has 27 singular points.

Using the canonical resolution in [2], we obtain a non-singular model M of A/G. The smooth algebraic variety M is *simply connected. The canonical bundle of M is trivial,* hence a fortiori $\kappa(M)=0$ and $H^1(M,\Theta)=0$ where Θ is the sheaf of germs of holomorphic vector fields on M.

5. We note that the above theorems and propositions have counterparts in the category of complex spaces.

References

[1] S. Iitaka: On D-dimensions of algebraic varieties (to appear in Journal of the Mathematical Society of Japan).
[2] K. Ueno: On fibre spaces of normally polarized abelian varieties of dimension 2. I. Singular fibres of the first kind (to appear in Journal of the Faculty of Science, University of Tokyo, Section I A).