37. *Dynamical System with Ergodic Partitions*

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Introduction. In this paper we give a sufficient condition for a dynamical system on a compact metric Lebesque space to be ergodic. The following argument is essentially described in the work of Sinai [2]. The difference is the consideration of "measurable set" in place of "k-dimensional submanifold".

0. Notations. We denote by $(M, \mathfrak{B}, \mu)$ a Lebesque space with σ-algebra $\mathfrak{B}$ and a measure μ; $\mu(M)=I$. We suppose that M is a compact metric space with distance $d(\ ,\ )$.

(M, T, μ) is a dynamical system, i.e., T is an automorphism of $(M, \mathfrak{B}, \mu)$.

$\mathfrak{N}$ denotes the trivial subalgebra consisting of sets of measure zero or of measure one, and $\mathfrak{S}(A_\alpha)$ the σ-algebra generated by the system of measurable sets $\{A_\alpha\}$.

$\mathfrak{S}|_A$ means the restriction of a σ-subalgebra $\mathfrak{S}$ to a measurable set A.

1. Expansive partitions and contractive partitions.

Definition 1.1. Let $\xi=\{C_\xi\}$ be a partition of M into measurable sets $\{C_\xi\}$. ξ is called to be *T-expansive* (*T-contractive*), if for two points $x, y \in M$ which belong to the same element C_ξ of ξ $d(T^n x, T^n y)$ $(d(T^{-n}x, T^{-n}y))$ converges to zero as $n \to \infty$.

Theorem 1.1. *Let ξ, η be two partitions of M. If one is T-expansive and the other is T-contractive then any T-invariant summable function is $\mathfrak{S}(\xi) \cap \mathfrak{S}(\eta)$-measurable.*

2. The partition of M which is not necessarily measurable may be measurable, if it is considered locally in some sense.

Definition 2.1. $\mathfrak{U}=\{U_k | k=1, 2, \cdots\}$ is called *a local basis* of M, if 1) $\mathfrak{S}(\mathfrak{U})=\mathfrak{B}$, 2) for any measurable set A such that $0<\mu(A)<1$ there exists some $U_k \in \mathfrak{U}$ which satisfies;

$$\mu(A \cap U_k) \cdot \mu(A^c \cap U_k) \neq 0$$

Definition 2.2. Let $\mathfrak{U}=\{U_k | k=1, 2, \cdots\}$ be a local basis of M. $\{(U_k, \xi_k) | k=1, 2, \cdots\}$ is called a *measurable fibre structure* (m.f.s.), if

1) ξ_k is a measurable partition of U_k,

2) for almost all $x \in U_k \cap U_l$, $C_{\xi_k}(x) \cap U_l = C_{\xi_l}(x) \cap U_k$, where $C_{\xi_k}(x)$ is an element of ξ_k which contains x.

An m.f.s. $\{(U_k, \xi_k) | k=1, 2, \cdots\}$ defines an equivalence relation $\sim$ and hence induces a partition of M. The relation $x \sim y$ for $x, y \in M$ means that there exist $U_{k_1}, \cdots, U_{k_j} \in \mathfrak{U}$ such that $x \in U_{k_1}$, $y \in U_{k_j}$ and

$C_{\xi_{k_i}} \cap U_{k_{i+1}} = C_{\xi_{k_{i+1}}} \cap U_{k_i}$ for $i=1, 2, \cdots, j-1$. Evidently $\sim$ is an equivalence relation of M.

Theorem 2.1. *Every equivalence class is a measurable set. And hence a partition of M is defined.*

We denote such a partition by ξ.

3. Two measurable fibre structures.

When two measurable fibre structures $\{(U_k, \xi_k) | k=1, 2, \cdots\}$, $\{(V_l, \eta_l) | l=1, 2, \cdots\}$ are given, there exist two measures μ_1, μ_2 defined on $\mathfrak{S}(\eta_l) | C_{\xi_k}(x)$, $x \in U_k \cap V_l$;

$$\mu_1(A) \equiv \mu(A / C_{\xi_k}(x)), \ A \in \mathfrak{S}(\eta_l) | C_{\xi_k}(x).$$

(i.e. μ_1 is a canonical measure with respect to the measurable partition ξ_k of U_k.)

$$\mu_2(A) \equiv \mu(\tilde{A}), \qquad \begin{array}{l} \tilde{A} \in \mathfrak{S}(\eta_l) \\ A = \tilde{A} \cap C_{\xi_k}(x) \end{array}$$

Definition 3.1. An m.f.s. $\{(U_k, \xi_k) | k=1, 2, \cdots\}$ is *absolutely continuous* with respect to another m.f.s. $\{(V_l, \eta_l) | l=1, 2, \cdots\}$, if

1) the measure μ_1, μ_2 are equivalent,

2) $\mu(U_k \cap V_l) > 0$ implies $\mu_2(C_{\xi_k}(x) \cap V_l) > 0$.

Theorem 3.1. *If an m.f.s. $\{(U_k, \xi_k) | k=1, 2, \cdots\}$ is absolutely continuous with respect to another m.f.s. $\{(V_l, \eta_l) | l=1, 2, \cdots\}$ then $\mathfrak{S}(\xi) \cap \mathfrak{S}(\eta) = \mathfrak{N}$.*

4. We obtain the following main theorem by means of Theorem 1.1 and Theorem 3.1.

Theorem 4.1. *Let $\{(U_k, \xi_k) | k=1, 2, \cdots\}$, $\{(V_l, \eta_l) | l=1, 2, \cdots\}$ be two m.f.s.'s. If one is absolutely continuous with respect to the other, and if one of the partitions ξ and η is T-expansive and the other is T-contractive then T is ergodic.*

5. Example.

Baker's transformation. M is a unit square; $M = \{(x, y) | x, y \in [0, 1]\}$, $d\mu = dxdy$.

$$T; (x, y) \to \begin{cases} \left(2x, \frac{1}{2}y\right), & \text{if } 0 < x < \frac{1}{2},\ 0 < y \leqq 1 \\ \left(2x-1, \frac{1}{2}y + \frac{1}{2}\right), & \text{if } \frac{1}{2} \leqq x \leqq 1,\ 0 < y \leqq 1 \\ (x, y), & \text{if } x=0 \text{ or } y=0 \end{cases}$$

$$\xi = \{x \times (0, 1]; 0 < x \leqq 1\} \cup \{(x=0)\} \cup \{(y=0)\}.$$

$$\eta = \{(0, 1) \times y; 0 < y \leqq 1\} \cup \{(x=0)\} \cup \{(y=0)\}.$$

ξ is T-expansive and η is T-contractive. Evidently, $\mathfrak{S}(\xi \wedge \eta) = \mathfrak{N}$. Applying Theorem 1.1, we conclude that T is ergodic.

References

[1] V. A. Rochlin: On the fundamental ideas of measure theory. I. Amer. Math. Soc. Trans., **10**, 1–54 (1962).

[2] Ja. G. Sinai: Dynamical systems with countably multiple Lebesque spectrum. II. Amer. Math. Soc. Trans., **68**, 34–88 (1968).