17. *Operators Satisfying the Growth Condition* (G_1)

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(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1971)

1. This note is motivated by the following theorem by I. H. Sheth.

Theorem 1 [6]. *Let $T=UR$, $R=(T^*T)^{1/2}$ be an invertible hyponormal operator such that U is cramped, then $0 \notin \overline{W(T)}$.*

The purpose of this note is to prove a generalization of Theorem 1 to the case of operators satisfying the growth condition (G_1). The technique of [6] actually proves the following theorem.

Theorem 2. *Let $T=UR$, $R=(T^*T)^{1/2}$ be an invertible operator such that T satisfies (G_1) and U is cramped, then $0 \notin \overline{W(T)}$.*

In the case of normal operator, this was proved by Berberian [1]. Durszt [2] constructed an invertible operator T such that the unitary operator $U=T(T^*T)^{-1}$ is cramped and $0 \in \overline{W(T)}$.

2. In the following, an operator means a bounded linear operator on a Hilbert space. Let T be an operator on H, $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T respectively. Let conv $\sigma(T)$ be the (automatically closed) convex hull of $\sigma(T)$. The numerical range $W(T)$ is the set $W(T)=\{(Tx, x) : x \in H, \|x\|=1\}$. We write $\overline{W(T)}$ the closure of $W(T)$. T satisfies the condition (G_1) if

$$(G_1) \qquad \|(T-\alpha I)^{-1}\| \leq 1/d(\alpha, \sigma(T))$$

for all $\alpha \notin \sigma(T)$, where $d(\alpha, \sigma(T))$ is the distance from α to $\sigma(T)$. A unitary operator U is cramped if $\sigma(U) \subset \{e^{i\theta} : \theta_0 < \theta < \theta_0 + \pi\}$.

If T is hyponormal, T satisfies Condition (G_1). In fact, in this case $(T-\alpha I)^{-1}(\alpha \notin \sigma(T))$ is also hyponormal, hence

$$\|(T-\alpha I)^{-1}\| = 1/\inf \{|\lambda-\alpha| : \lambda \varepsilon \sigma(T)\} = 1/d(\alpha, \sigma(T)).$$

Let X be a compact convex set of the complex plane. A point $\lambda \in X$ is bare if there is a circle through λ such that no points of X lie outside this circle.

3. To prove Theorem 2, we use the following facts which are stated as lemmas.

Lemma 1. *If U is unitary, U is cramped if and only if $0 \notin \overline{W(U)}$.*

Proof. See [1 : Lemma 3].

Lemma 2. *Let T be an operator which satisfies Condition* (G_1), *then every bare point λ of $\overline{W(T)}$ is contained in $\sigma_a(T)$ and has the following property: $Tx_n - \lambda x_n \to 0$ $(n \to \infty)$ if and only if $T^*x_n - \bar{\lambda} x_n \to 0$ $(n \to \infty)$ for a sequence $\{x_n\}$ of unit vectors.*

Proof. Since T satisfies Condition (G_1), $\overline{W(T)}=\text{conv }\sigma(T)$ by [4: Theorem 2]. Thus every bare point λ of $\overline{W(T)}$ is contained in $\sigma_a(T)$. The second assertion follows from [5: Theorem 1].

Lemma 3. *Let X be a compact convex set of the complex plane and let B_X be the set of all bare points of X. Then X is the closed convex hull of B_X.*

Proof. See [4: Lemma 3].

Proof of Theorem 2. Suppose that $0\in\overline{W(T)}$, then $0\in\text{conv }\sigma(T)$, because Condition ($G_1$) implies $\overline{W(T)}=\text{conv }\sigma(T)$. Let $\varepsilon>0$ be given. By Lemma 3, there exist bare points $\alpha_1,\alpha_2\cdots,\alpha_r$ of $\overline{W(T)}$ and real numbers $a_1,a_2,\cdots,a_r$ such that

$$a_k\geq 0\ (k=1,2,\cdots,r);\quad \sum_{k=1}^r a_k=1;\quad \left|\sum_{k=1}^r a_k\alpha_k\right|<\varepsilon.$$

By Lemma 2, for each $k=1,2,\cdots,r$ there exists a sequence $\{x_n^{(k)}\}$ of unit vectors such that

$$\|Tx_n^{(k)}-\alpha_k x_n^{(k)}\|\to 0\qquad(n\to\infty)$$

and

$$\|T^*x_n^{(k)}-\bar{\alpha}_k x_n^{(k)}\|\to 0\qquad(n\to\infty).$$

Since

$$T^*T-|\alpha_k|^2=T^*(T-\alpha_k)+\alpha_k(T^*-\bar{\alpha}_k),$$

we see that

$$\|T^*Tx_n^{(k)}-|\alpha_k|^2x_n^{(k)}\|\to 0\qquad(n\to\infty),$$

for each $k=1,2,\cdots,r$. Thus for every polynomial $p(\lambda)$,

$$\|p(T^*T)x_n^{(k)}-p(|\alpha_k|^2)x_n^{(k)}\|\to 0\qquad(n\to\infty).$$

Since $R=(T^*T)^{1/2}$ is a strong limit of a sequence of polynomials of T^*T, there exists an integer $N>0$ such that

$$\|Rx_n^{(k)}-|\alpha_k|\,x_n^{(k)}\|<\varepsilon\qquad(n>N)$$

for each $k=1,2,\cdots,r$. Note that
$\inf\{|\alpha|:\alpha\in B_{\overline{W(T)}}\}\geq\gamma>0$, for $0\notin\sigma(T)$. Since

$$T-\alpha_k I=U(R-|\alpha_n|I)+|\alpha_k|\left(U-\frac{\alpha_k}{|\alpha_k|}I\right),$$

$$\begin{aligned}|\alpha_k|\|Ux_n^{(k)}-\frac{\alpha_k}{|\alpha_k|}x_n^{(k)}\|&\\ &\leq\|Tx_n^{(k)}-\alpha_k x_n^{(k)}\|+\|Rx_n^{(k)}-|\alpha_k|\,x_n^{(k)}\|\\ &<2\varepsilon\qquad(n>N)\end{aligned}$$

for each $k=1,2,\cdots,r$. Since $\varepsilon>0$ is arbitrary, this shows that $\frac{\alpha_k}{|\alpha_k|}\in\sigma(U)$. Let $b_j=\frac{a_j\alpha_j}{\sum_{k=1}^n a_k|\alpha_k|}$
for $j=1,2,\cdots,r$, then

$$\sum_{j=1}^r a_j\alpha_j=\left(\sum_{k=1}^n a_k|\alpha_k|\right)\sum_{j=1}^r b_j\frac{\alpha_j}{|\alpha_j|};$$

$$b_j\geq 0\ (j=1,2,\cdots,r):\ \sum_{j=1}^r b_j=1.$$

Since $\sum_{k=1}^{n} a_k|\alpha_k| \geq \gamma > 0$, we have

$$\left|\sum_{j=1}^{r} b_j \frac{\alpha_j}{|\alpha_j|}\right| < \varepsilon/\gamma.$$

Since $\varepsilon > 0$ is arbitrary, $0 \in \operatorname{conv} \sigma(U) = \overline{W(U)}$. This is a contradiction, for U is cramped. Hence $0 \notin \overline{W(T)}$.

Let T be an operator such that

$$\|T - \alpha I\| = \sup\{|\lambda - \alpha| : \lambda \in \sigma(T)\}$$

for all α, then $W(T) = \operatorname{conv} \sigma(T)$, but the second assertion of Lemma 2 is open in this case.

In conclusion we mention a result by Williams [7]. He proved that if $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, then $\sigma(T)$ is real. This result implies that if $\overline{W(T)} = \operatorname{conv} \sigma(T)$ and $S^{-1}TS = T^*$ with $0 \notin \overline{W(S)}$, then T is self-adjoint. In fact, $W(T)$ is real in this case.

References

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