8. A Note for Knots and Flows on 3-manifolds

By Gikō IKEGAMI*) and Dale ROLFSEN**)

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H. Seifert shows in [1] (Satz 11) that for any torus knot k in the 3-sphere S^3 there is a flow on S^3 with k as an orbit, and conversely, that if a homotopy 3-sphere Σ^3 admits a flow on it so that all orbits are closed then $\Sigma^3=S^3$ and each orbit is a torus knot.

Here, we consider the following question: For any knot k in S^3 does there exist a non-singular flow on S^3 having k as an orbit, allowing for the flow having non-closed orbits? In this paper, we give an affirmative answer to this question.

Manifolds and maps, etc in this paper are assumed to be smooth (C^∞-) ones. A *flow* on a manifold M is a 1-parameter group of transformations $\phi: R\times M\to M$ (R, the real numbers). $x\in M$ is said to be a *singular point* if $\phi(t,x)=x$ for all $t\in R$. ϕ is said to be *non-singular* if there is no singular point. An *orbit* of ϕ passing x is a subset $\{\phi(t,x)\mid t\in R\}$. If there is $t\neq 0$ such that $\phi(t,x)=x$, the orbit is said to be *closed*.

Let f be a map of S^1 into a space M and $p: R\to S^1$ be the usual universal covering defined by $t\mapsto e^{2\pi ti}$, then we shall denote $f\circ p=\bar{f}$.

Theorem. *Let M be an orientable closed 3-manifold and $f: S^1\to M$ be an embedding. Then, there exist a flow $\phi: R\times M\to M$ and $x\in M$ such that $\phi(t,x)=\bar{f}(t)$ for all $t\in R$.*

Proof. Denote the tangent bundle of M by $T(M)$. Since, by [2] (Satz 21), M is parallelizable, we may assume $T(M)=M\times R^3$. Consider the $(R^3-\{0\})$-bundle $T(M)$, $\xi: M\times(R^3-\{0\})\to M$ over M associated to tangent bundle. We define a map $g: f(S^1)\to T(M)$ as follows: for $x\in f(S^1)$, $g(x)=d\bar{f}/dt(t)$ where t is any number such that $\bar{f}(t)=x$. g is well-defined. Since f is an embedding, g is a cross-section of ξ over $f(S^1)$. We will extend g to a cross-section of ξ over M.

We may take a tubular neighborhood U of $f(S^1)$ coordinated as follows;

$$U=\{(x,r,\theta)\mid x\in f(S^1),\quad 0\leqq r\leqq 1,\quad 0\leqq\theta<2\pi\}$$

with

$$(x,0,\theta)=(x,0,0)\quad \text{for all } x \text{ and } \theta.$$

Since $\pi_1(R^3-\{0\})\cong\pi_1(S^2)=0$, we have a homotopy F of $q\circ g$ as follows, where q is the projection into the second factor $M\times(R^3-\{0\})\to R^3-\{0\}$:

*) Kōbe University.
**) The University of British Columbia.

$$F: f(S^1)\times[0,1]\to R^3-\{0\},$$
$$F(x,0)=q\circ g(x)$$
$$F(x,1)=*, \text{ a fixed point of } R^3-\{0\}.$$

Next, we define a map $G: M\to R^3-\{0\}$, as follows.

$$G(x,r,\theta)=\begin{cases} q\circ g(x) & \text{if} \quad 0\leqq r<\dfrac{1}{2} \\ F(x,2r-1) & \text{if} \quad \dfrac{1}{2}\leqq r\leqq 1 \end{cases}$$

$$G(y)=* \qquad \text{if} \quad y\notin U.$$

G is continuous. By an approximation keeping fixed on $f(S^1)$, we may make G a smooth map $\tilde{G}$. If we put $(y,\tilde{G}(y))=\tilde{g}(y)$, $\tilde{g}: M\to M\times(R^3-\{0\})$ is a cross-section of ξ, and also, it is an extension of g.

We may assume that $\tilde{g}$ is a non-zero vector field on M extending g. The flow, obtained by integrating $\tilde{g}$, is the desired one. This proves the Theorem.

Let l be an embedding $\{S_1^1\cup\cdots\cup S_n^1\}\to M$, where S_i^1 is a circle and $S_1^1\cup\cdots\cup S_n^1$ is the disjoint union, then we call l a *link* in M and each $l(S_i^1)$ a *component* of the link.

Corollary. *For any link l of an orientable closed 3-manifold M, there exists a non-singular flow ϕ of M such that each component of l coincides with a certain orbit of ϕ.*

The proof is similar to the one of the Theorem.

Remark. There is a well-known Seifert's Conjecture which states that every non-singular flow on S^3 has a closed orbit. The Theorem states that if we solve the Seifert's Conjecture we must take it into consideration that any knot may come out as the closed orbit.

References

[1] H. Seifert: Topologie dreidimensionaler gefaserter Räume. Acta Math., **60**, 147–238 (1933).

[2] E. Stiefel: Richtungsfelder und Fernparallelismus in n-dimensionaler Mannigffaltigkeiten. Comm. Math. Helv., **8**, 305–353 (1936).