27. *Localization Theorem in Hyperbolic Mixed Problems*

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Introduction. Recently Atiyah, Bott and Gårding [1] have studied some interesting properties (behavior near the wave fronts, supports, singular supports and lacunas, etc.) and structures of fundamental solutions of hyperbolic differential operators with constant coefficients. It seems that some of their methods can be applicable to the study of Riemann or Green's functions (kernels) for hyperbolic mixed problems in a quarter-space. The properties of such Riemann functions are less investigated. For example there are Deakin [2], Duff [3] Hersh [4], etc. In this note we present one of properties which can be easily proved, more precisely "localization theorem" corresponding to one in the free space-time case. The idea of localizing fundamental solutions is due to Hörmander [6].

1. Riemann or Green's functions. Let $\boldsymbol{R}^n$ denote the n-dimensional euclidean space and $\boldsymbol{\Xi}^n$ its complex dual space, we shall write $x'=(x_1,\cdots,x_{n-1})$, $x''=(x_2,\cdots,x_n)$ for the coordinate $x=(x_1,\cdots,x_n)$ in $\boldsymbol{R}^n$ and $\xi'=(\xi_1,\cdots,\xi_{n-1})$, $\xi''=(\xi_2,\cdots,\xi_n)$ for the dual coordinate $\xi=(\xi_1,\cdots,\xi_n)$. The variable x_1 will play the role of "time", the variables $x_2,\cdots,x_n$ will play the role of "space". We shall also denote by $\boldsymbol{R}^n_+$ the half-space $\{x=(x',x_n)\in \boldsymbol{R}^n, x_n>0\}$. For differentiation we will use the symbol $D=1/i\cdot\partial/\partial x$.

Let $P=P(\xi)$ be a hyperbolic polynomial of n variables ξ with respect to $\vartheta=(1,0,\cdots,0)\in \mathrm{Re}\,\boldsymbol{\Xi}^n$ in the sense of Gårding, i.e. $P_m(\vartheta)\neq 0$ and $P(\xi+t\vartheta)\neq 0$ when ξ is real and $\mathrm{Im}\,t$ is less than some fixed number where P_m denotes the principal part of P. We consider the following mixed initial-boundary value problem for the hyperbolic operator $P(D)$ in a quarter-space

(1) $$P(D)u(x)=f(x),\quad x\in \boldsymbol{R}^n_+,\quad x_1>0,$$

(2) $$(D_1^k u)(0,x'')=0,\quad k=0,1,\cdots,m-1,\quad x_n>0,$$

(3) $$B_j(D)u(x)\Big|_{x_n=0}=0,\quad j=1,\cdots,l,\quad x_1>0.$$

Here $B_j(D)$ are boundary operators with order m_j. The number of boundary conditions will be determined later on.

We assume that the coefficients of ξ_n^m in $P(\xi)$ and $\xi_n^{m_j}$ in $B_j(\xi)$ are different from zero, i.e. that the hyperplane $x_n=0$ is non-characteristic for $P(D)$ and $B_j(D)$. We shall construct the Riemann function $G(x,y)$ which describes the propagation of waves produced by a unit impulse given at position $y=(0, y_2, \cdots, y_n) \in \boldsymbol{R}_+^n$ and at time $x_1=0$ in a medium whose states are governed by (1)–(3). Let $\operatorname{Re} A$ be the real hypersurface $\{\xi \in \operatorname{Re} \Xi^n, P_m(\xi)=0\}$. If we denote by $\Gamma=\Gamma(A, \vartheta)$ the component of $\operatorname{Re} \Xi^n - \operatorname{Re} A$ which contains ϑ, Γ is an open convex cone. The dual cone $K=K(A, \vartheta)=\{x \in \boldsymbol{R}^n ; x \cdot \xi \geqslant 0, \xi \in \Gamma\}$ is called the propagation cone. As is well known, a hyperbolic operator $P(D)$ has a fundamental solution $E(x)$ satisfying $P(D)E(x)=\delta(x)$ and having support in the propagation cone K. The fundamental solution is defined as an inverse Fourier-Laplace transform of P^{-1} in the form.

$$E(x)=(2\pi)^{-n}\int_{\operatorname{Re}\Xi^n} e^{ix\cdot(\xi+i\eta)}P(\xi+i\eta)^{-1}\,d\xi, \tag{4}$$

where $\eta \in s\vartheta - \Gamma$ with s large enough and $\vartheta=(1, 0, \cdots, 0)$.

Let $y=(0, y_2, \cdots, y_n)$ be a point of $\boldsymbol{R}_+^n$. Then the fundamental solution $E(x-y)$ describes the incident or primary propagation of waves due to a point source $\delta(x''-y'')$. The distribution which describes the propagation of secondary waves reflected from a plane boundary $x_n=0$ subject to the boundary conditions $B_j(D)$ is given as a solution of the problem:

$$P(D_x)F(x,y)=0, \quad x \in \boldsymbol{R}_+^n, \quad x_1>0, \tag{5}$$

$$(D_1^k F)(0, x'', y)=0, \quad k=0,1,\cdots,m-1, \quad x=(0,x'') \in \boldsymbol{R}_+^n, \tag{6}$$

$$B_j(D_x)F(x,y)\Big|_{x_n=0}=B_j(D_x)E(x-y)\Big|_{x_n=0}, \quad j=1,\cdots,l. \tag{7}$$

Then $G(x,y)=E(x-y)-F(x,y)$ satisfies the equations $P(D_x)G(x,y) =\delta(x-y), x \in \boldsymbol{R}_+^n, x_1>0$ and $B_j(D_x)G(x,y)\Big|_{x_n=0}=0, j=1,\cdots,l$. Hence one may consider $G(x,y)$ as Riemann function for the hyperbolic mixed problem (1)–(3). In order to construct $F(x,y)$ we start by formal discussions. From (4) we have

$$\mathcal{F}-\mathcal{L}_{x'}[B_j(D_x)E(x-y)\Big|_{x_n=0}]=(2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-iy(\xi+i\eta)}\frac{B_j(\xi+i\eta)}{P(\xi+i\eta)}\,d\xi_n$$

where $\xi \in \operatorname{Re}\Xi^n$ and $\eta \in -s\vartheta-\Gamma$ with s large enough. Taking thus formally partial Fourier-Laplace transforms in (5) and (7) with respect to $x'=(x_1, \cdots, x_{n-1})$, we obtain a boundary value problem for ordinary differential equation with parameters:

$$P(\xi'+i\eta', D_n)\hat{F}(\xi'+i\eta', x_n, y)=0, \quad x_n>0, \tag{8}$$

$$\begin{aligned} &B_j(\xi'+i\eta', D_n)\hat{F}(\xi'+i\eta', x_n, y)\Big|_{x_n=0} \\ &\quad=(2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-iy(\xi+i\eta)}\frac{B_j(\xi+i\eta)}{P(\xi+i\eta)}\,d\xi_n. \end{aligned} \tag{9}$$

2. Algebraic considerations. In order to give an explicit representation of temperate solutions of this problem, it is necessary to study the roots of the algebraic equation in λ with parameters $\xi' : P(\xi', \lambda)=0$. First we note

Lemma ([1], p. 132). *By the hyperbolicity of P, there exists a* *non-negative number* s_0 *such that* $P(\xi+s\vartheta)\neq 0$ *when* ξ *is real and* $|\mathrm{Im}\, s|>s_0$. *Then we have*

(10) $$P(\xi+t\eta+s\vartheta)\neq 0$$

when ξ *is real,* $\eta\in\Gamma$, $\mathrm{Im}\, t\leqslant 0$ *and* $\mathrm{Im}\, s<-s_0$ (resp. $\mathrm{Im}\, t\geqslant 0$ *and* $\mathrm{Im}\, s>s_0$). *For* P_m, *we have* $P_m(\xi+t\eta+s\vartheta)\neq 0$ *when* ξ *is real,* $\eta\in\Gamma$, $\mathrm{Im}\, t\leqslant 0$ *and* $\mathrm{Im}\, s<0$ (resp. $\mathrm{Im}\, t\geqslant 0$ *and* $\mathrm{Im}\, s>0$).

From (10) it follows that the roots of $P(\xi',\lambda)=0$ are never real when $\xi'\in\mathrm{Re}\,\Xi^{n-1}-is\vartheta'-i\Gamma_0$ (resp. $\mathrm{Re}\,\Xi^{n-1}+is\vartheta'+i\Gamma_0$) with s large enough where $\Gamma_0=\{\eta\,;(\eta',0)\in\Gamma\}$. Since the roots of $P(\xi',\lambda)=0$ are (multivalued) continuous as functions of ξ'(note $P_m(0,1)\neq 0$), this implies that the number of roots with positive imaginary part, counted according to multiplicity, is constant when $\xi'\in\mathrm{Re}\,\Xi^{n-1}-is\vartheta'-i\Gamma_0$ (resp. $\mathrm{Re}\,\Xi^{n-1}+is\vartheta'+i\Gamma_0$). This number l determines the number of boundary conditions required in the general theory of hyperbolic mixed problems. When $\xi'\in\mathrm{Re}\,\Xi^{n-1}-is\vartheta'-i\Gamma_0$, we denote by $\lambda_k(\xi'), k=1, \cdots, l(\lambda_k(\xi'), k=l+1,\cdots,m)$ the roots of $P(\xi',\lambda)=0$ with positive (negative) imaginary part and set

$$P^+=P^+(\xi',\lambda)=P_m(0,1)\prod_{k=1}^{l}(\lambda-\lambda_k(\xi')).$$

Since the coefficients of the polynomial P^+ in λ are elementary symmetric functions of $\lambda_1(\xi'),\cdots,\lambda_l(\xi')$, they are analytic functions of ξ' in $\mathrm{Re}\,\Xi^{n-1}-is\vartheta'-i\Gamma_0$ as is well known in analytic function theory.[1] We shall also denote by $\mu_k(\xi')$ $k=1,\cdots,m$ the roots of $P_m(\xi',\mu)=0$. Since $t^{-m}P(t\xi',t\mu)\to P_m(\xi',\mu)$ as $t\to\infty$ it follows that, with suitable labelling,

(11) $$t^{-1}\lambda_k(t\xi')\to\mu_k(\xi'),\ k=1,\cdots,m \text{ as } t\to\infty.$$

By the preceding lemma, the number of the roots of $P_m(\xi',\mu)=0$ with positive (resp. negative) imaginary part, is constant when $\xi'\in\mathrm{Re}\,\Xi^{n-1}-i\Gamma_0$. By the relation (11) this number is equal to l (resp. $m-l$), moreover

$$\mathrm{Im}\,\mu_k(\xi')>0,\ k=1,\cdots,l \text{ and } \mathrm{Im}\,\mu_k(\xi')<0,\ k=l+1,\cdots,m.$$

Let $D(P^+)(\xi')$ and $D(P_m^+)(\xi')$ denote the discriminants of $P^+(\xi',\lambda)=0$ in λ and $P_m^+(\xi',\mu)=P_m(0,1)\prod_{k=1}^{l}(\mu-\mu_k(\xi'))=0$ in μ respectively. Then $D(P^+)(\xi')$ can be continuously extended to $\mathrm{Re}\,\Xi^{n-1}-i(\Gamma_0\cup\{0\})$. We now define Lopatinski determinant for the system $\{P,B_j\}$ by

$$R(\xi')=R(P^+,B_1,\cdots,B_l)=\det(B_j(\xi',\lambda_k(\xi')))/\prod_{i>k}(\lambda_i(\xi')-\lambda_k(\xi')).$$

Likewise, Lopatinski determinant for $\{P_m,B_j^0\}$ is defined by

$$R^0(\xi')=R(P_m^+,B_1^0,\cdots,B_l^0)=\det(B_j^0(\xi',\mu_k(\xi')))/\prod_{i>k}(\mu_i(\xi')-\mu_k(\xi')).$$

where $B_j^0(D)$ denotes the principal part of $B_j(D)$ for each j. Since $\det(B_j(\xi',\lambda_k))/\prod_{i>k}(\lambda_i-\lambda_k)$ is a symmetric function of the variables $\lambda_1,\cdots,\lambda_l$ with polynomial coefficients of ξ', $R(\xi')$ can be expressed as

1) For example, see the proof of Weierstrass preparation theorem in Goursat's book: Cours d'Analyse Mathematique. II.

a polynomial (with poly. coeff. of ξ') in the coefficient of $P^+(\xi', \lambda)$.[2] Hence $R(\xi')$ is an analytic function of ξ' in $\mathrm{Re}\,\Xi^{n-1}-is\vartheta'-i\Gamma_0(s>s_1)$. By the same reason $R^0(\xi')$ is an analytic function of ξ' in $\mathrm{Re}\,\Xi^{n-1}-i\Gamma_0$ and extended continuously to $\mathrm{Re}\,\Xi^{n-1}-i(\Gamma_0\cup\{0\})$. From now on we shall assume the Lopatinski condition:

(12) $R(\xi')\neq 0$ when $\xi'\in \mathrm{Re}\,\Xi^{n-1}-is\vartheta'-i\Gamma_0$ with s large enough.[3]

Further we assume $R^0(\xi')\neq 0$ when $0\neq\xi'\in\{\xi'\,;\,\xi\in \mathrm{Re}\,A\}$. Let $R_j(x_n,\xi')=R(P^+,B_1,\cdots,B_{j-1},e^{i\lambda x_n},B_{j+1},\cdots,B_l)$ be the determinant obtained by replacing in $R(P^+,B_1,\cdots,B_l)$ j-row with the vector $(e^{i\lambda_1(\xi')x_n},\cdots e^{i\lambda_l(\xi')x_n})$, when $D(P^+)(\xi')\neq 0$ and by continuity otherwise (cf. Hörmander [5]).

3. Reflected Riemann functions for $\{P, B_j\}$. Localization.

Under the preliminaries in the preceding section, a unique temperate solution of (8)–(9) for each $\xi'\in\mathrm{Re}\,\Xi^{n-1}$ and $\eta\in -s\vartheta-\Gamma$ with s large enough, has the representation

$$\hat{F}(\xi'+i\eta',x_n,y)=(2\pi)^{-1}\sum_{j=1}^{l}\frac{R_j(x_n,\xi'+i\eta')}{R(\xi'+i\eta')}\int_{-\infty}^{+\infty}e^{-iy(\xi+i\eta)}\frac{B_j(\xi+i\eta)}{P(\xi+i\eta)}d\xi_n.$$

Thus we have formally

$$F(x,y)=(2\pi)^{-n}\int_{\mathrm{Re}\,\Xi^{n-1}}\sum_{j=1}^{l}\frac{R_j(x_n,\xi'+i\eta')}{R(\xi'+i\eta')}e^{i(x'-y')(\xi'+i\eta')}\times\left\{\int_{-\infty}^{+\infty}e^{-iy_n(\xi_n+i\eta_n)}\frac{B_j(\xi+i\eta)}{P(\xi+i\eta)}d\xi_n\right\}d\xi'. \tag{13}$$

where ξ is real and $\eta\in -s\vartheta-\Gamma$ with s large enough and $(\eta',0)\in\Gamma$. If $m_j<m$, (13) can be interpreted in the distribution sense with respect to x. However, in order to establish our localization theorem, $F(x,y)$ has to be interpreted in the sense of distribution with respect to $(x,y)\in\overline{\boldsymbol{R}^n_+}\times\boldsymbol{R}^n_+$ as

$$\begin{aligned}&\langle F(x,y),\check{\varphi}(x')\psi(x_n)\otimes g(y')h(y_n)\rangle\\&=(2\pi)^{-n}\int\sum_{j=1}^{l}\frac{R_j(x_n,\xi'+i\eta')}{R(\xi'+i\eta')}e^{-iy'(\xi'+i\eta')}\hat{\varphi}(\xi'+i\eta')\psi(x_n)g(y')\\&\quad\times\left\{\int e^{-iy_n(\xi_n+i\eta_n)}\frac{B_j(\xi+i\eta)}{P(\xi+i\eta)}h(y_n)d\xi_n dy_n\right\}d\xi' dx_n dy',\end{aligned} \tag{14}$$

where $\varphi,g\in C_0^\infty(\boldsymbol{R}^{n-1})$, $\psi\in C_0^\infty(\overline{\boldsymbol{R}^1_+})$ and $h\in C_0^\infty(\boldsymbol{R}^1_+)$. We now define by (14) the secondary or reflected Riemann kernel for the system $\{P,B_j\}$.

According to Atiyah-Bott-Gårding [1], we introduce the notion of localization of polynomials

Definition. *Let $P(\xi)$ be a polynomial of degree $m\geqslant 0$ and develop $t^mP(t^{-1}\xi+\zeta)$ in ascending power of t*

$$t^mP(t^{-1}\xi+\zeta)=t^pP_\xi(\zeta)+O(t^{p+1}),$$

2) Let $f(\lambda_1,\cdots,\lambda_l)$ be a symmetric polynomial of $(\lambda_1,\cdots,\lambda_l)$. Then f can be represented as a polynomial of elementary symmetric functions and the coefficients are linear functions (with integer coefficients) in the coefficients of f.

3) In order to establish our localization theorem, it is sufficient to assume that $R(\xi')\neq 0$ when $\xi'\in\mathrm{Re}\,\Xi^{n-1}-is\,\vartheta'$ with s large enough. But the assumption (12) will be necessary for the study of supports of reflected Riemann functions.

where $P_\xi(\zeta)$ is the first coefficient that does not vanish identically in ζ. The number $p=m_\xi(P)$ is called the multiplicity of ξ relative to P, the polynomial $\zeta\to P_\xi(\zeta)$, the localization of P at ξ.

If $P_m(\xi)\neq 0$, then $P_\xi(\zeta)=P_m(\xi)$ is a non-zero constant. When P is hyperbolic, we are interested in localizations at real points ξ such that $P_m(\xi)=0$. When $\xi\in \operatorname{Re} A$ and $D(P^+)(\xi')\neq 0$, we call ξ *regular point* of $\operatorname{Re} A$. Denote by $\Delta(\xi')$ the determinant $(B_j^0(\xi',\mu_k(\xi')))$ and by $\Delta_{jk}(\xi')$ its (j,k)-cofactor. We set with δ small positive

$$\alpha_k(\xi')=\frac{1}{2\pi i}\oint_{|z-\mu_k(\xi')|=\delta} z\left\{P_m(\xi',z)\frac{\partial P_{m-1}}{\partial z}(\xi',z)-\frac{\partial P_m}{\partial z}(\xi',z)P_{m-1}(\xi',z)\right\}/\{P(\xi',z)\}^2dz$$

and define the localizations of $F(x,y)$ at a regular point ξ by

$$F_{\xi,k}(x,y)=(2\pi)^{-n}\sum_{j=1}^{l}\frac{\Delta_{jk}(\xi')B_j^0(\xi)}{\Delta(\xi')}e^{i\alpha_k(\xi')x_n}\times\int_{\operatorname{Re}\Xi^{n-1}} e^{i(x'-y'+x_n\operatorname{grad}_{\xi'}\mu_k(\xi')\cdot(\zeta'+i\eta')}\left\{\int_{-\infty}^{+\infty}\frac{e^{-iy_n(\zeta_n+i\eta_n)}}{P_\xi(\zeta+i\eta)}d\zeta_n\right\}d\zeta',$$

if $\mu_k(\xi')$ is real, $(1\leqslant k\leqslant l)$ and $F_\xi(x,y)=0$ otherwise. Then we have

Localization theorem. *Let ξ be a regular point of $\operatorname{Re} A$ and $p=m_\xi(P)$ the multiplicity of ξ relative to P. Then if $\mu_k(\xi')$ is real,*

$$\boldsymbol{R}\ni t\to t^{p-m}e^{-it\{(x'-y')\xi'+\mu_k(\xi')x_n-\xi_n y_n\}}F(x,y)\to F_{\xi,k}(x,y)$$

in the sense of distribution with respect to $(x,y)\in \boldsymbol{R}_+^n\times\boldsymbol{R}_+^n$ and

$$\xi\neq 0\Rightarrow \operatorname{supp}_{(x,y)}F_{\xi,k}\subset \operatorname{sing\,supp}_{(x,y)}F.$$

Moreover

$$\operatorname{supp}_{(x,y)}F_{\xi,k}\subset\{(x,y)\in\boldsymbol{R}_+^n\times\boldsymbol{R}_+^n;\quad y=(0,y_1,\cdots,y_n),\ [x'-y'+x_n\operatorname{grad}_{\xi'}\mu_k(\xi')]\cdot\eta'-y_n\eta_n\geqslant 0,\ \eta\in\Gamma_\xi\},$$

where $\Gamma_\xi=\Gamma(A_\xi,\vartheta)$ and $\operatorname{Re} A_\xi=\{\zeta\in\operatorname{Re}\Xi^n, P_{m\xi}(\zeta)=0\}$.

The detailed proof will be given in a forthcoming paper.

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