121. *Paracompactifications of M-spaces*

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By a space we shall always mean a completely regular Hausdorff space unless otherwise specified.

1. Let X be a space with a uniformity Φ agreeing with the topology of X; that is, Φ is a family of open coverings of X satisfying conditions (a) to (c) below, where for coverings $\mathfrak{U}$ and $\mathfrak{V}$ of X we mean by $\mathfrak{U}<\mathfrak{V}$ that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$.

(a) If $\mathfrak{U}, \mathfrak{V} \in \Phi$, then there exists $\mathfrak{W} \in \Phi$ such that $\mathfrak{U}<\mathfrak{W}$ and $\mathfrak{V}<\mathfrak{W}$.

(b) If $\mathfrak{U} \in \Phi$, there is $\mathfrak{V} \in \Phi$ which is a star-refinement of $\mathfrak{U}$.

(c) $\{\mathrm{St}(x, \mathfrak{U}) \mid \mathfrak{U} \in \Phi\}$ is a basis of neighborhoods at each point x of X.

Let $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ be the totality of those normal sequences which consist of open coverings of X contained in Φ. Let $\Phi_\lambda=\{\mathfrak{U}_{\lambda i} \mid i=1,2,\cdots\}$, where $\mathfrak{U}_{\lambda i}$ is a star-refinement of $\mathfrak{U}_{\lambda, i-1}$ for $i=2,3,\cdots$. As in [1], we denote by (X, Φ_λ) the topological space obtained from X by taking $\{\mathrm{St}(x, \mathfrak{U}_{\lambda i}) \mid i=1,2,\cdots\}$ as a basis of neighborhoods at each point x of X. Let X/Φ_λ be the quotient space obtained from (X, Φ_λ) by defining those two points x and y equivalent for which $y \in \mathrm{St}(x, \mathfrak{U}_{\lambda i})$, for $i=1,2,\cdots$. Then there is a canonical map $\varphi_\lambda: X \to X/\Phi_\lambda$ which is continuous, and X/Φ_λ is metrizable.

Now we shall define a partial order in $\{\Phi_\lambda \mid \lambda \in \Lambda\}$. If each member of Φ_λ has a refinement in Φ_μ, we write $\Phi_\lambda<\Phi_\mu$. Then, if $\Phi_\lambda<\Phi_\mu$, there exists a continuous map $\varphi_\lambda^\mu: X/\Phi_\mu \to X/\Phi_\lambda$ such that $\varphi_\lambda=\varphi_\lambda^\mu \circ \varphi_\mu$, and $\{X/\Phi_\lambda; \varphi_\lambda^\mu\}$ is an inverse system of metrizable spaces. Let us set

$$\mu_\Phi(X)=\varprojlim X/\Phi_\lambda.$$

For any point x of X, let us put $\varphi(x)=\{\varphi_\lambda(x)\}$. Then $\varphi: X \to \mu_\Phi(X)$ is a homeomorphism into.

In case every Cauchy family $\{C_\gamma\}$ of X with respect to Φ which has the countable intersection property is non-vanishing (that is, $\cap \bar{C}_\gamma \neq \phi$), we say that X is *weakly complete* with respect to Φ.

Theorem 1. *The map $\varphi: X \to \mu_\Phi(X)$ is onto if and only if X is weakly complete with respect to Φ.*

In case Φ is the finest uniformity (that is, Φ consists of all normal open coverings of X), we write $\mu(X)$ instead of $\mu_\Phi(X)$. In this case we have the following theorems.

Theorem 2. *$\mu(X)$ is the completion of X with respect to its finest uniformity and for any continuous map $f: X \to Y$ there is a continuous map $\mu(f): \mu(X) \to \mu(Y)$ so that μ is a covariant functor from the category of spaces to the category of topologically complete spaces.*

Here a space is called *topologically complete* if it is complete with respect to its finest uniformity.

Theorem 3. *$\mu(X)$ is characterized as a space Y with the following properties:*

(a) *Y is a topologically complete space containing X as a dense subspace.*

(b) *Any continuous map f from X into a metric space T can be extended to a continuous map from Y into T.*

2. Now, let X be an M-space throughout this section (cf. [1]). Then, by definition, there is a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X satisfying the condition (M):

(M) If $\{K_i\}$ is a decreasing sequence of non-empty closed sets of X such that $K_i \subset \mathrm{St}(x, \mathfrak{U}_i)$ for each i and for some point x of X, then $\cap K_i \neq \phi$.

Let $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ be the totality of all normal sequences of open coverings of X and $\{\Phi_\lambda \mid \lambda \in \Lambda'\}$ the set of all normal sequences Φ_λ satisfying Condition (M). Then $\{\Phi_\lambda \mid \lambda \in \Lambda'\}$ is cofinal in $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ and we have

$$\mu(X) = \varprojlim \{X/\Phi_\lambda\,;\, \lambda \in \Lambda'\}.$$

Moreover $\varphi_\lambda^\mu: X/\Phi_\mu \to X/\Phi_\lambda$ is a perfect map if $\Phi_\lambda < \Phi_\mu$ and $\lambda, \mu \in \Lambda'$. In general, we have

Theorem 4. *If $\{X_\lambda\,;\, \varphi_\lambda^\mu\}$ is an inverse system of spaces such that each φ_λ^μ is a perfect map, then the projection from $\varprojlim \{X_\lambda\,;\, \varphi_\lambda^\mu\}$ to X_λ is a perfect map for each λ.*

Hence we have the first part of the following theorem.

Theorem 5. *Let X be an M-space. Then $\mu(X)$ is a paracompact M-space, and moreover $\mu(X) = \beta(f)^{-1}(T)$ for any quasi-perfect map f from X onto a metric space T, where $\beta(f): \beta(X) \to \beta(T)$ is the Stone extension of f.*

Thus we may call $\mu(X)$ the *paracompactification of X*.

Theorem 6. *If $f: X \to Y$ is a quasi-perfect map, where X, Y are M-spaces, then $\mu(f): \mu(X) \to \mu(Y)$ is a perfect map.*

Theorem 7. *Let X be an M-space. Then X admits a quasi-perfect map from X onto a separable (resp. locally compact or complete) metric space if and only if $\mu(X)$ is Lindelöf (resp. locally compact or a G_δ in its Stone-Čech compactification).*

Theorem 8. *Let f be a quasi-perfect map from an M-space onto an M-space Y. If X admits a quasi-perfect map from X onto a* *separable (resp. locally compact or complete) metric space, so does Y.*

3. In this section we are concerned with the product formula $\mu(X\times Y)=\mu(X)\times\mu(Y)$, which, however, does not hold in general.

Theorem 9. *For any space X and a locally compact, paracompact space Y we have $\mu(X\times Y)=\mu(X)\times\mu(Y)$.*

Theorem 10. *Let $X\times Y$ be an M-space. Then the following conditions are equivalent.*

(a) $\mu(X\times Y)=\mu(X)\times\mu(Y)$.

(b) *There exist quasi-perfect maps $\varphi: X\to S$, $\psi: Y\to T$ with S, T metrizable such that the product map $\varphi\times\psi: X\times Y\to S\times T$ is a quasi-perfect map.*

(c) *If K and L are any countably compact closed subsets of X and Y respectively, then $K\times L$ is countably compact.*

4. As an application of Theorem 10 we have the following theorem, where, following M. Katětov (cf. [2]), we define dim X for a not necessarily normal space X by dim $\beta(X)$ ($\beta(X)$ being the Stone-Čech compactification of X).

Theorem 11. *Let X be an M-space and Y a metric space or a locally compact paracompact space. Then* $\dim(X\times Y)\leqq\dim X+\dim Y$.

It seems that Theorem 11 is the first result which assures the validity of the product theorem on dimension for the case of $X\times Y$ being not necessarily normal.

The proofs of the theorems stated above and the details will be published elsewhere.

References

[1] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365-382 (1964).

[2] L. Gillman and M. Jerrison: Rings of Continuous Functions. Van Nostrand, Princetion (1960).