113. An Application of Serre-Grothendieck Duality Theorem to Local Cohomology

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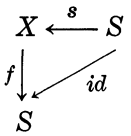
The purpose of this note is to prove the following theorem using the Serre-Grothendieck duality theorem and to derive two formulae from it. These formulae show that some algebro-geometric notions can be expressed with local cohomology.

Theorem. *Let X and S be locally noetherian preschemes of finite Krull dimension. Let $f: X \to S$ be a proper, smooth morphism of relative dimension n, and let $s: S \to X$ be a f-section. We identify the image of s with S and denote by $\mathcal{J}$ the sheaf of ideals in $\mathcal{O}_X$ defining the closed subprescheme S. Denote by $\omega_{X/S}$ the sheaf of n-th differential forms on X relative to f. Then for every coherent sheaf $\mathcal{F}$ on X and for every integer $p \geq 0$, there exists a functorial isomorphism*

$$(1) \qquad f_*(\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^n_S(\omega_{X/S}))) \simeq \varinjlim_{k\geq 0} \mathcal{E}xt^p_{\mathcal{O}_S}(f_*(\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}^{k+1})), \mathcal{O}_S).$$



Remark 1. We can eliminate the regularity condition for f using derived functors. We can treat even more general cases ([N]). The proof becomes, however, so complicated in spite of little merit of generalization.

Proposition (SGA II, exposé VI, Theorem 2.3). *Let X be a locally noetherian prescheme and let Y be a closed subprescheme of X defined by a coherent sheaf of ideals $\mathcal{J}$. Then for every coherent sheaf $\mathcal{F}$ on X and for every quasi-coherent sheaf $\mathcal{G}$ on X, there is a spectral sequence*

$$(2) \qquad \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^q_Y(\mathcal{G})) \Rightarrow \varinjlim_{k\geq 0} \mathcal{E}xt^{p+q}(\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}^{k+1}), \mathcal{G}).$$

Lemma. *Under the same hypothesis of the theorem, let $\mathcal{G}$ be a sheaf of abelian groups on X with support in S. Then for all $p>0$,*

$$R^p f_*(\mathcal{G}) = 0.$$

Proof. On the category of sheaves of abelian groups whose supports are in S, it is obvious that the direct image functor f_* is the same as the restriction functor s^* under the above hypothesis. Since s^* is an exact functor and the canonical flasque resolution of $\mathcal{G}$ can be given in the above category, the conclusion follows.

Proof of Theorem. First, we prove a corollary of Serre-Grothendieck duality, which says that for every quasi-coherent sheaf $\mathcal{G}$ on X, there is a functorial isomorphism in $D_{qc}(S)$, (We use freely the notation and the terminology of [RD] here.)

$$\mathcal{R}f_*\mathcal{R}\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \omega_{X/S}[n]) \xrightarrow{\sim} \mathcal{R}\,\mathcal{H}om_{\mathcal{O}_S}(\mathcal{R}f_*(\mathcal{G}), \mathcal{O}_S).$$

Assume further that the support of $\mathcal{G}$ is in S. Then by the lemma, when we take the cohomology on both sides, we have a functorial isomorphism for each $p\geq 0$, as a corollary of duality theorem,

$$f_*(\mathcal{E}xt^{p+n}_{\mathcal{O}_X}(\mathcal{G}, \omega_{X/S})) \xrightarrow{\sim} \mathcal{E}xt^{p}_{\mathcal{O}_S}(f_*(\mathcal{G}), \mathcal{O}_S).$$

Especially if we substitute $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1})$ for $\mathcal{G}$,

$$f_*(\mathcal{E}xt^{p+n}_{\mathcal{O}_X}(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1}), \omega_{X/S})) \xrightarrow{\sim} \mathcal{E}xt^{p}_{\mathcal{O}_S}(f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1})), \mathcal{O}_S).$$

Since both sides form inductive systems for k and the functor f_* commutes with the inductive limit in the category of abelian group sheaves on locally noetherian spaces, we have a functorial isomorphism for every $p\geq 0$

$$(3)\qquad f_*(\varinjlim_{k\geq 0} \mathcal{E}xt^{p+n}_{\mathcal{O}_X}(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1}), \omega_{X/S})) \xrightarrow{\sim} \varinjlim_{k\geq 0} \mathcal{E}xt^{p}_{\mathcal{O}_S}(f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1})), \mathcal{O}_S).$$

Since s is a regular immersion of codimension n and $\omega_{X/S}$ is an invertible sheaf by the hypothesis, one can easily derive from [LC. 3.8] that for all $p\neq n$

$$\mathcal{H}^p_S(\omega_{X/S})=0.$$

Then the spectral sequence (2) degenerates when $Y=S$ and $\mathcal{G}=\omega_{X/S}$ in the proposition and we have a functorial isomorphism

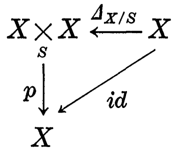
$$(4)\qquad \mathcal{E}xt^{p}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^n_S(\omega_{X/S})) \xrightarrow{\sim} \varinjlim_{k\geq 0} \mathcal{E}xt^{p+n}_{\mathcal{O}_X}(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} (\mathcal{O}_X/\mathcal{J}^{k+1}), \omega_{X/S}).$$

The theorem follows from the two functorial isomorphisms (3) and (4).

Corollary 1. *Let X, S, f and $\omega_{X/S}$ be the same as in the theorem. Denote by p and q the first and the second projections from $X\underset{S}{\times}X$ to X respectively, and denote by $\Delta_{X/S}$ the diagonal immersion from X to $X\underset{S}{\times}X$. Let $\mathcal{D}iff_{X/S}$ be the sheaf of differential operators on X relative to f* ([EGA, IV, 10.8.7]). *Then, there is an isomorphism*

$$(5)\qquad \mathcal{H}^n_{\Delta_{X/S}}(q^*\omega_{X/S}) \xrightarrow{\sim} \mathcal{D}iff_{X/S}.$$

Proof. We use the theorem for the following diagram, $\mathcal{F}=\mathcal{O}_{X\underset{S}{\times}X}$ and $p=0$.



The conclusion follows when we notice that $\omega_{X \underset{S}{\times} X/X} \xrightarrow{\sim} q^*\omega_{X/S}$ and p_* is isomorphic to the restriction to X in this case.

Remark 2. When X and S are algebraic varieties over the complex number field, the formula (5) can be interpreted as follows. Let $\mathcal{O}_{hol,X}$ be the sheaf of holomorphic functions on X and let $\omega_{hol,X/S}$ be the sheaf of holomorphic n-forms on X relative to f. In the theory of Sato's hyperfunctions the sheaf $\mathcal{L}=\mathcal{H}^n_{\Delta X/S}(q^*\omega_{hol,X/S})$ is used as the sheaf of "generalized" differential operators on X relative to f ([S]). According to the canonical homomorphism (which is injective in fact) induced by $\omega_{X/S}\to\omega_{hol,X/S}$,

$$\mathcal{O}_{hol,X} \underset{\mathcal{O}_X}{\otimes} \mathcal{H}^n_{\Delta X/S}(q^*\omega_{X/S})\to\mathcal{H}^n_{\Delta X/S}(q^*\omega_{hol,X/S}),$$

the formula (5) implies that the "algebraic" part of $\mathcal{L}$ consists of the "usual" differential operators.

Corollary 2. *Let X be a locally noetherian prescheme of finite Krull dimension and let $\mathcal{L}$ be an invertible sheaf on X. Denote by $\mathcal{L}^\vee$ the dual of $\mathcal{L}$. Let $V=V(\mathcal{L}^\vee)$ be the line bundle associated to $\mathcal{L}^\vee$* ([EGA, II, 1.7.8]). *Denote by $\omega_{V/X}$ the sheaf of the first differential forms on V relative to X. Let s be the zero section from X to V and identify $s(X)$ with X. Then there is an isomorphism*

(6) $$\mathcal{H}^1_X(\omega_{V/X}) \xrightarrow{\sim} \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}.$$

Proof. Let $\mathcal{S}$ be the graded sheaf $(\bigoplus_{n\geq 0} \mathcal{L}^{\otimes(-n)}) \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X[T]$, where T is an indeterminate and the grade of $\mathcal{S}$ is given canonically and let $\bar{V}$ be Proj $(\mathcal{S})$. Denote by p the projection from $\bar{V}$ to X.

$$\begin{array}{ccc} \bar{V} & \xleftarrow{\ s\ } & X \\ {\scriptstyle p}\big\downarrow & \swarrow {\scriptstyle id} & \\ X & & \end{array}$$

Using the theorem in this case, we have

(7) $$\mathcal{H}^1_X(\omega_{\bar{V}/X}) \xrightarrow{\sim} \varinjlim_{k\geq 0} \mathcal{H}om_{\mathcal{O}_X}(p_*(\mathcal{O}_{\bar{V}}/\mathcal{J}^{k+1}), \mathcal{O}_X),$$

where $\mathcal{J}$ is the ideal sheaf defining X in $\bar{V}$.

Since $\bar{V}$ contains V canonically and $\bar{V}-V$ and $s(X)$ have no intersection, we can write V instead of $\bar{V}$ in (7). It is easy to see by [EGA, II, 8.10.1] the right hand of (7) is isomorphic to $\bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$.

Remark 3. If X is a prescheme over a field of characteristic zero, we can easily generalize the corollary (2) as follows. Let $\mathcal{L}$ be a locally free sheaf of rank n on X and $\omega_{V/X}$ be the sheaf of n-th relative differential forms. The other situation is the same. Then there is an isomorphism

(6)′ $$\mathcal{H}^n_X(\omega_{V/X}) \xrightarrow{\sim} \mathbf{S}(\mathcal{L}),$$

where $\mathbf{S}(\mathcal{L})$ denotes the sheaf of symmetric algebras of $\mathcal{L}$.

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