32. *L^p-theory of Pseudo-differential Operators*

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Introduction. The L^2-theory of pseudo-differential operators has been studied in many papers, but we know very few papers which are concerned with L^p-theory. We say $g(x, \xi) \in S^m_{\rho,\delta}$, $0<\rho \leqq 1$, $0 \leqq \delta$, when $g(x, \xi) \in C^\infty(R^n_x \times R^n_\xi)$ and for any α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_x \partial^\beta_\xi g(x, \xi)| \leqq C_{\alpha,\beta}\langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|}$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ are multi-indices whose elements are non-negative integers, $\langle \xi \rangle=(1+|\xi|^2)^{\frac{1}{2}}$, and $\partial_{x_j}=\partial/\partial x_j$, $\partial_{\xi_j}=\partial/\partial \xi_j$, $j=1, \cdots, n$,

$$\partial^\alpha_x=\partial^{\alpha_1}_{x_1}\cdots\partial^{\alpha_n}_{x_n},\ \partial^\beta_\xi=\partial^{\beta_1}_{\xi_1}\cdots\partial^{\beta_n}_{\xi_n},\ |\alpha|=\alpha_1+\cdots+\alpha_n,$$

$|\beta|=\beta_1+\cdots+\beta_n$. For a pseudo-differential operator defined by the symbol of class $S^m_{\rho,\delta}$, the L^2-boundedness of the form $\|g(X, D_x)u\|_s \leqq C\|u\|_{m+s}$ was proved by Hörmander [2] and Kumano-go [4] in the case $0 \leqq \delta < \rho \leqq 1$.

In the present paper we shall study the general L^p-theory for pseudo-differential operators of class $S^m_{1,\delta}$ in the case: $0 \leqq \delta < 1$ and $1<p<\infty$. Recently for operators of class $S^0_{1,\delta}$, Kagan [3] proved the L^p-boundedness: $\|p(X, D_x)u\|_{L^p} \leqq C\|u\|_{L^p}$ for $1<p \leqq 2$. Applying the theory in Kumano-go [5], we first prove the inequality $\|g(X, D_x)u\|_{p,s} \leqq C\|u\|_{p,m+s}$ for any real s and $1<p<\infty$ (which solves a problem of Hörmander in [2], p. 163, for the typical case $\rho=1$), and prove the theorems: the generalized Poincaré inequality, the invariance of the space $H_{p,s}$ under coordinate transformation and the a priori estimate for elliptic operators.

1. Definitions and fundamental lemmas.

We shall use the following notations:

$$\mathcal{S}=\{u(x) \in C^\infty(R^n)\,;\ \lim_{|x|\to\infty} |x|^m|\partial^\alpha_x u(x)|=0 \text{ for any } m \text{ and } \alpha\}.$$

$\mathcal{S}'$ denotes the dual space of $\mathcal{S}$. For $u \in \mathcal{S}$, we define the Fourier transform of u by $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$. For any real s we define an operator $\langle D_x\rangle^s$: $\mathcal{S}\to\mathcal{S}$ by

$$\langle D_x\rangle^s u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}\langle\xi\rangle^s\hat{u}(\xi)d\xi.$$

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We define the norm $\|u\|_{p,s}$ by

$$\|u\|_{p,s}=\left\{\int|\langle D_x\rangle^s u(x)|^p dx\right\}^{1/p}.$$

The operator $\langle D_x\rangle^s$: $\mathcal{S}\to\mathcal{S}$ can be uniquely extended to the operator $\langle D_x\rangle^s$: $\mathcal{S}'\to\mathcal{S}'$ by

$$\langle\langle D_x\rangle^s u, v\rangle=\langle u, \langle D_x\rangle^s v\rangle \quad \text{for} \quad u\in\mathcal{S}',\ v\in\mathcal{S}.$$

Definition 1.1. *For $1<p<\infty$ and $-\infty<s<\infty$ we define the Sobolev space $H_{p,s}$ by $H_{p,s}=\{u\in\mathcal{S}'; \langle D_x\rangle^s u\in L^p(R^n)\}=\{u\in\mathcal{S}'; u=\langle D_x\rangle^{-s}u_0$ for some $u_0\in L^p(R^n)\}$.*

By the definition we can easily see that $H_{p,s}$ is a Banach space provided with the norm $\|u\|_{p,s}$, and $\mathcal{S}$ is dense in $H_{p,s}$.

Definition 1.2. *For $g(x,\xi)\in S^m_{\rho,\delta}$ we define an operator $g(X,D_x)$ by $g(X,D_x)u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}g(x,\xi)\hat{u}(\xi)d\xi$ for $u\in\mathcal{S}$.*

It is clear that $g(X,D_x)$: $\mathcal{S}\to\mathcal{S}$ is linear. In what follows we assume that $0\leqq\delta<1$ and $1<p<\infty$. For $g(x,\xi)\in S^m_{1,\delta}$ we use a notation $|g|_l=|g|_{l,m}$ defined by

$$|g|_{l,m}=\operatorname*{Max}_{|\alpha+\beta|\leqq l}\ \sup_{x,\xi}\{|\partial_x^\alpha\partial_\xi^\beta g(x,\xi)|\langle\xi\rangle^{-(m+\delta|\alpha|-|\beta|)}\}<\infty.$$

Lemma 1.1 (*Kagan* [3]). *Assume that $1<p\leqq 2$. For any $g(x,\xi)\in S^0_{1,\delta}$ there exists a constant C such that*

$$\|g(X,D_x)u\|_{p,0}\leqq C\|u\|_{p,0} \quad \text{for} \quad u\in\mathcal{S}, \tag{1.1}$$

where C depends only on p and $|g|_{l,0}$ for sufficiently large l.

Lemma 1.2 (*Kumano-go* [5]). i) *For two symbols $g_j(x,\xi)\in S^{m_j}_{1,\delta}$, $j=1, 2$, there exists a symbol $g(x,\xi)\in S^{m_1+m_2}_{1,\delta}$ of the form $g(x,\xi)=g_1(x,\xi)g_2(x,\xi)+g'(x,\xi)$ where $g'(x,\xi)\in S^{m_1+m_2-(1-\delta)}_{1,\delta}$ such that $g(X,D_x)=p_1(X,D_x)p_2(X,D_x)$.*
ii) *For a symbol $g(x,\xi)\in S^m_{1,\delta}$ there exists a symbol $g^*(x,\xi)\in S^m_{1,\delta}$ of the form $g^*(x,\xi)=\overline{g(x,\xi)}+g'(x,\xi)$ where $g'(x,\xi)\in S^{m-(1-\delta)}_{1,\delta}$ such that $(g(X,D_x)u,v)=(u,g^*(X,D_x)v)$ for any $u,v\in\mathcal{S}$, where we used the notation*

$$(u,v)=\int u(x)\overline{v(x)}dx \quad \text{for any} \quad u,v\in\mathcal{S}.$$

Theorem 1.1. *For $g(x,\xi)\in S^m_{1,\delta}$ and real s, there exists a constant $C=C(m, |g|_{l,m}, s)$ such that*

$$\|g(X,D_x)u\|_{p,s}\leqq C\|u\|_{p,m+s} \quad \text{for} \quad u\in\mathcal{S}. \tag{1.2}$$

Remark. Set $s_0=n(1/p-1/q)$ for $1<p\leqq q<\infty$. By the Hardy-Littlewood-Sobolev estimates of potentials we have $\|v\|_{q,-s_0}\leqq C_{p,q}\|v\|_{p,0}$, $v\in\mathcal{S}$, with a constant $C_{p,q}$. Then, by Theorem 1.1, we get $\|g(X,D_x)u\|_{q,-s_0}\leqq C\|u\|_{p,0}$, $u\in\mathcal{S}$, for $g(x,\xi)\in S^0_{1,\delta}$. This means that Hörmander's problem in [2], p. 163, holds for $\rho=1$.

Proof 1°. The case $m=0$ and $s=0$. In this case in view of Lemma 1.1, we may assume that $p>2$. Let $p'=p/(p-1)$, then $1<p'<2$. By ii) of Lemma 1.2 there is a symbol $g^*(x,\xi)\in S^0_{1,\delta}$ such that

$(g(X, D_x)u, v)=(u, g^*(X, D_x)v)$. Then, by Lemma 1.1 and Hölder's inequality we have

$$|(g(X, D_x)u, v)|=|(u, g^*(X, D_x)v)|$$
$$\leqq \|u\|_{p,0}\, \|g^*(X, D_x)v\|_{p',0} \leqq C\|u\|_{p,0}\, \|v\|_{p',0}.$$

Therefore by the duality theorem we get $g(X, D_x)u \in L^p$ and

$$\|g(X, D_x)u\|_{p,0} \leqq C\|u\|_{p,0}.$$

2^o. The general case. Since $\langle\xi\rangle^s \in S^s_{1,0}$, by i) of Lemma 1.2 there is a symbol $g_s(x, \xi) \in S^{m+s}_{1,\delta}$ such that $g_s(X, D_x)=\langle D_x\rangle^s g(X, D_x)$. Therefore we have

$$\|g(X, D_x)u\|_{p,s}=\|g_s(X, D_x)u\|_{p,0}$$
$$=\|(g_s(X, D_x)\langle D_x\rangle^{-(m+s)})(\langle D_x\rangle^{m+s}u)\|_{p,0}.$$

Since $p_s(x, \xi)\langle\xi\rangle^{-(m+s)} \in S^0_{1,\delta}$, by 1^o we obtain (1.2). Q.E.D.

2. The properties of the space $H_{p,s}$ and Poincaré's lemma.

Proposition 2.1. *If $s_1 \geqq s_2$, then $H_{p,s_1} \subset H_{p,s_2}$ and*

(2.1) $\|u\|_{p,s_2} \leqq C(s_1, s_2, p)\, \|u\|_{p,s_1}$ *for* $u \in H_{p,s_1}$ (*c.f.* [1], p. 120).

Proof. Noting $\langle\xi\rangle^{-(s_1-s_2)} \in S^0_{1,0}$, by Theorem 1.1 we have

$$\|u\|_{p,s_2}=\|\langle D_x\rangle^{s_2}u\|_{p,0}=\|\langle D_x\rangle^{-(s_1-s_2)}(\langle D_x\rangle^{s_1}u)\|_{p,0}$$
$$\leqq C\|\langle D_x\rangle^{s_1}u\|_{p,0}=C\|u\|_{p,s_1} \quad \text{for} \quad u \in \mathcal{S}.$$

Since $\mathcal{S}$ is dence in H_{p,s_1}, this means (2.1). Q.E.D.

Theorem 2.1 (*Poincaré's lemma*). *For any $1<p<\infty$ and any real $s>0$ there exists a constant C such that*

(2.2) $\|u\|_{p,0} \leqq Cd^s\|u\|_{p,s}$ *for* $u \in C_0^\infty(|x|<d)$

where C depends only on p and s and is independent of $d>0$.

Proof. We may only prove the theorem for $0<d<1$, since (2.2) is clear for $d \geqq 1$ by means of (2.1). Let $\psi(\xi) \in C_0^\infty(R^n)$ such that $\psi(\xi)=1$ for $|\xi| \leqq 1/2$ and $\psi(\xi)=0$ for $|\xi| \geqq 1$, and let $\psi_{d,\varepsilon}(\xi)=\psi(d\varepsilon^{-1}\xi)$ where ε is a sufficiently small positive number to be fixed later. We define $u_1(x)$, $u_2(x)$ by $\hat{u}_1(\xi)=\psi_{d,\varepsilon}(\xi)\hat{u}(\xi)$ and $\hat{u}_2(\xi)=\{1-\psi_{d,\varepsilon}(\xi)\}\hat{u}(\xi)$, respectively. Then we have $u(x)=u_1(x)+u_2(x)$. Set $g(\xi)=g_{d,\varepsilon}(\xi)=d^{-s}\langle\xi\rangle^{-s}\{1-\psi_{d,\varepsilon}(\xi)\}$. Then,

$$\partial_\xi^\alpha g(\xi)$$
$$=d^{-s}\sum_{\substack{\alpha'+\alpha''=\alpha\\ \alpha''\neq 0}} C_{\alpha,\alpha'}\partial_\xi^{\alpha'}\langle\xi\rangle^{-s}\cdot\left(\frac{d}{\varepsilon}\right)^{|\alpha''|}\psi^{(\alpha'')}\left(\frac{d}{\varepsilon}\xi\right)+d^{-s}\partial_\xi^\alpha\langle\xi\rangle^{-s}\cdot\{1-\psi_{d,\varepsilon}(\xi)\}.$$

Since $d\langle\xi\rangle \geqq \varepsilon/2$ on the support of $\{1-\psi_{d,\varepsilon}(\xi)\}$, and $\varepsilon/2 \leqq d\langle\xi\rangle \leqq C_o$ on the support of $\psi^{(\alpha'')}(d\varepsilon^{-1}\xi)$ where C_o is independent of $0<d<1$, we have $|\partial_\xi^\alpha g(\xi)| \leqq C_{\alpha,\varepsilon}\langle\xi\rangle^{-|\alpha|}$. Hence by Theorem 1.1 we have

$$\|u_2\|_{p,0}=d^s\, \|g(D_x)\langle D_x\rangle^s u\|_{p,0}$$
$$\leqq d^s C_{1,\varepsilon}\|\langle D_x\rangle^s u\|_{p,0}=d^s C_{1,\varepsilon}\|u\|_{p,s}$$

where $C_{1,\varepsilon}$ is independent of d. We can write

$$u_1(x)=\int\hat{\psi}_{d,\varepsilon}(y-x)u(y)dy=\int\left(\frac{\varepsilon}{d}\right)^n\hat{\psi}\left(\frac{\varepsilon}{d}(y-x)\right)u(y)dy.$$

We can see easily that $|\hat{\psi}(z)| \leqq C_2$ and $\|\hat{\psi}_{d,\varepsilon}\|_{L^1}=\|\hat{\psi}\|_{L^1}=C_3$ where C_2 and C_3 are independent of d and ε. Therefore,

$$|u_1(x)|^p \leqq \left(\int_{|y|<d} |\hat{\psi}_{d,\varepsilon}(x-y)|\,dy\right)^{p/p'} \left(\int |\hat{\psi}_{d,\varepsilon}(x-y)|\,|u(y)|^p dy\right)$$
$$\leqq C_{n,p} C_2^{p/p'} \varepsilon^{np/p'} \int |\hat{\psi}_{d,\varepsilon}(x-y)|\,|u(y)|^p dy.$$

Hence $\|u_1\|_{p,0}^p \leqq C_{n,p} C_2^{p/p'} C_3 \varepsilon^{np/p'} \|u\|_{p,0}^p$, and taking $\varepsilon>0$ sufficiently small, we get $\|u_1\|_{p,0} \leqq \frac{1}{2}\|u\|_{p,0}$. Then, we have

$$\|u\|_{p,0} \leqq \|u_1\|_{p,0} + \|u_2\|_{p,0} \leqq \frac{1}{2}\|u\|_{p,0} + C_1 d^s \|u\|_{p,s},$$

and get (2.2) for $C=2C_1$. Q.E.D.

Corollary. *Let $s'>s>0$ and $d>0$. Then there exists a constant $C=C(s',s,p,n)$, which is independent of $d>0$, such that*

(2.3) $$\|u\|_{p,s} \leqq C d^{s'-s} \|u\|_{p,s'} \quad \text{for} \quad u \in C_0^\infty(|x|<d).$$

Next we consider a C^∞-coordinate transformation $x(y)\colon R_y^n \to R_x^n$ such that

(2.4) $$\partial_{y_j} x_k(y) \in \mathscr{B}_y,\ j,k=1,\cdots,n,\ C^{-1} \leqq |\det(\partial_y x(y))| \leqq C$$

for a constant $C>0$ where $\partial_y x(y) = (\partial_{y_j} x_k(y))$ is the Jacobian matrix and $\det(\partial_y x(y))$ denotes its determinant. For $u \in \mathcal{S}$ we put $w(y)=u(x(y))$.

Lemma 2.1 (*Kumano-go* [5]). *For $\langle\xi\rangle^m \in S_{1,0}^m$ there exists a symbol $h(y,\eta) \in S_{1,0}^m$ such that $h(Y,D_y)w(y) = (\langle D_x\rangle^m u)(x(y))$.*

Theorem 2.2. *The space $H_{p,s}$ is invariant under the coordinate transformation satisfying (2.4) in the sense: $u(x) \in H_{p,s,x}$ if and only if $w(y)=u(x(y)) \in H_{p,s,y}$. More precisely there exist symbols $h(y,\eta) \in S_{1,0}^{-s}$ and $g(x,\xi) \in S_{1,0}^{-s}$ such that $w(y)=h(Y,D_y)w_0(y)$ for $w_0(y)=u_0(x(y))$ if $u=\langle D_x\rangle^{-s}u_0$ for $u_0 \in L^p$ and $u(x)=g(X,D_x)u_0(x)$ for $u_0(x)=w_0(y(x))$ if $w=\langle D_y\rangle^{-s}w_0$ for $w_0 \in L^p$.*

Remark. Theorem 2.2 was shown by Lions-Magenes [6] for the more general case, but here we give another proof which is simple and concrete.

Proof. We may only prove the inequality:

$$C^{-1}\|u\|_{p,s,x} \leqq \|w\|_{p,s,y} \leqq C\|u\|_{p,s,x} \quad \text{for} \quad u(x) \in \mathcal{S}_x,\ w(y)=u(x(y)) \in \mathcal{S}_y.$$

By Lemma 2.1 there is a symbol $h(y,\eta) \in S_{1,0,y}^{-s}$ such that $w(y)=u(x(y)) = (\langle D_x\rangle^{-s}u_0)(x(y)) = h(Y,D_y)w_0(y)$ where $w_0(y)=u_0(x(y))$. Therefore, by Theorem 1.1,

$$\|w(y)\|_{p,s,y} = \|h(Y,D_x)w_0(y)\|_{p,s,y} \leqq C_1\|w_0\|_{p,0,y}$$
$$\leqq C_2\|u_0\|_{p,0,x} = C_2\|u\|_{p,s,x}.$$

By the same way we have $\|u\|_{p,s,x} \leqq C\|w\|_{p,s,y}$. Q.E.D.

3. The a priori estimate for elliptic operators.

Lemma 3.1 (*Kumano-go* [5]). *Let $g(x,\xi) \in S_{1,\delta}^m$. Then, for any real s there exists a constant C_s such that*

(3.1) $$\|g(X,D_x)u\|_{2,s} \leqq |g|_{0,m}\|u\|_{2,m+s} + C_s\|u\|_{2,m+s-(1-\delta)/2}.$$

Lemma 3.2. *For $g(x,\xi) \in S_{1,\delta}^m$ there exist constants $C_p^{(1)}$ and $C_p^{(2)}$ such that $\lim_{p\to 2} C_p^{(1)} = |g|_{0,m}$ and*

(3.2) $$\|g(X,D_x)u\|_{p,s} \leqq C_p^{(1)}\|u\|_{p\ m+s} + C_p^{(2)}\|u\|_{p,m+s-(1-\delta)}.$$

Proof. Let $\psi(\xi)\in C^\infty$ such that $\psi(\xi)=0$ for $|\xi|\leqq 1$, $\psi(\xi)=1$ for $|\xi|\geqq 2$ and $0\leqq\psi(\xi)\leqq 1$, and set $\psi_k(\xi)=\psi(\xi/k)$, $k=1,2,\cdots$. Then, by Lemma 3.1 and Plancherel's formula we have

$$\|g(X,D_x)\psi_k(D_x)u\|_{2,s} \leqq \{|g|_{0,m}+C_s\sup_\xi(|\psi_k(\xi)|\langle\xi\rangle^{-(1-\delta)/2})\}\|u\|_{2,m+s}.$$

Therefore for any $\varepsilon>0$ there exists k_ε such that

$$\|g(X,D_x)\psi_{k_\varepsilon}(D_x)u\|_{2,s}\leqq(|g|_{0,m}+\varepsilon)\|u\|_{2,m+s}.$$

Then, by Theorem 1.1 and the interpolation theorem of Riesz-Thorin (see [7]), we get $\|g(X,D_x)\psi_k(D_x)u\|_{p,s}\leqq C_p\|u\|_{p,m+s}$, where $\lim_{p\to 2} C_p=|g|_{0,m}+\varepsilon$. Using the fact $g(x,\xi)(1-\psi_k(\xi))\in S^{-\infty}(=\bigcap_t S^t_{1,0})$ and taking a sequence $\varepsilon_1>\varepsilon_2>\cdots\to 0$, we get (3.2). Q.E.D.

Theorem 3.1. *Let $g(x,\xi)\in S^m_{1,\delta}$ satisfy $|g(x,\xi)|\geqq C_0\langle\xi\rangle^m$. Then there exist constants C_p, C'_p and $C^{(1)}_p$, $C^{(2)}_p$ such that*

$$\|u\|_{p,m+s}\leqq C_p\|g(X,D_x)u\|_{p,s}+C'_p\|u\|_{p,m+s-(1-\delta)}, \tag{3.3}$$

$$\|u\|_{p,m+s}\leqq C^{(1)}_p\|g(X,D_x)u\|_{p,s}+C^{(2)}_p\|u\|_{p,m+s-(1-\delta)}, \tag{3.4}$$

where C_p, C'_p are bounded when p is on any compact set of $(1,\infty)$ and $\lim_{p\to 2} C^{(1)}_p=C_0^{-1}$.

Proof. Setting $g_{-1}(x,\xi)=g(x,\xi)^{-1}$ $(\in S^{-m}_{1,\delta})$ we write

$$\begin{aligned}\|u\|_{p,m+s}\leqq &\|g_{-1}(X,D_x)\langle D_x\rangle^{m+s}g(X,D_x)u\|_{p,0}\\ &+\|g_{-1}(X,D_x)\{g(X,D_x)\langle D_x\rangle^{m+s}-\langle D_x\rangle^{m+s}g(X,D_x)\}u\|_{p,0}\\ &+\|\{1-g_{-1}(X,D_x)g(X,D_x)\}\langle D_x\rangle^{m+s}u\|_{p,0}.\end{aligned}$$

Then, using i) of Lemma 1.2 and Theorem 1.1 we can show that the second and third terms do not exceed $C'_p\|u\|_{p,m+s-(1-\delta)}$. As for the first term, by the assumption of $g(x,\xi)$ we get $g_{-1}(x,\xi)\langle\xi\rangle^{m+s}\in S^s_{1,\delta}$ and $\sup_{x,\xi}\{|g_{-1}(x,\xi)|\langle\xi\rangle^{m+s}\langle\xi\rangle^{-s}\}\leqq C_0^{-1}$. Therefore if we apply Theorem 1.1 to $g_{-1}(X,D_x)\langle D_x\rangle^{m+s}$, we have

$$\|g_{-1}(X,D_x)\langle D_x\rangle^{m+s}g(X,D_x)u\|_{p,0}\leqq C_p\|g(X,D_x)u\|_{p,s}.$$

Hence we get (3.3). By Lemma 3.2 and Theorem 1.1 we get (3.4) for $C^{(1)}_p$ such that $\lim_{p\to 2} C^{(1)}_p=C_0^{-1}$. Q.E.D.

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