33. *Continuous Affine Transformations of Locally Compact Totally Disconnected Groups*

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1. Introduction. In this paper the followings shall be proved. Let G be a locally compact totally disconnected non-discrete group and let T be a continuous automorphism of G. If there are two elements a and w in G such that $\{T(a)^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G then G is compact, where $T(a)$ is the continuous affine transformation of G defined by $T(a)(x)=a \cdot Tx$ for x in G. Next let G be a locally compact totally disconnected (not necessarily non-discrete) group and let T be a continuous automorphism of G such that there is an element w in G such that $\{T^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then G is compact, whence T. S. Wu's problem (see [1, p. 518] and also [6]) raised in 1967 concerning the study of topology of a locally compact group G which admits an ergodic continuous automorphism with respect to a Haar measure on G is solved affirmatively.

Recently M. Rajagopalan and B. Schreiber [4] have proved that if a locally compact group G has a continuous automorphism which is ergodic with respect to a Haar measure on G then G is compact. In their proof the property of Fourier-Stieltjes coefficients of idempotent measures on the torus $K=\{\exp(i\theta) \mid 0 \leqq \theta < 2\pi\}$ plays an important role. In studying their techniques of the proof I have been led to that the techniques can be applied to the arguments of continuous affine transformations.

2. Continuous affine transformations. Throughout this paper, T and $T(a)$ will be denoted a continuous automorphism of a locally compact group G and a continuous affine transformation of G induced by a in G and T, respectively.

Lemma 1. *Let H be a complex Hilbert space, let A be a bounded operator and U_1, U_2 unitary operators on H. Then for given ξ and η in H there is a complex regular measure μ on the 2-dimensional torus $K \times K$ whose Fourier-Stieltjes transform is given by*

$$\hat{\mu}(m,n)=\langle A U_1^m \xi, U_2^n \eta\rangle, \qquad -\infty < m, n < \infty.$$

Proof. Let ρ_1 and ρ_2 denote spectral measures on $[0, 2\pi)$ for U_1 and U_2, respectively. For ξ, η in H we have

$$\begin{aligned}\langle AU_1^m\xi, U_2^n\eta\rangle &= \left\langle A\int_0^{2\pi}\exp(im\theta)d\rho_1(\theta)\xi, \int_0^{2\pi}\exp(in\theta)d\rho_2(\theta)\eta\right\rangle \\ &= \left\langle\int_0^{2\pi}\exp(im\theta)dA\rho_1(\theta)\xi, \int_0^{2\pi}\exp(in\theta)d\rho_2(\theta)\eta\right\rangle \\ &= \int_0^{2\pi}\int_0^{2\pi}\exp i(m\theta_1-n\theta_2)d\langle A\rho_1(\theta_1)\xi, \rho_2(\theta_2)\eta\rangle \\ &= \int_0^{2\pi}\int_0^{2\pi}\exp i(m\theta_1-n\theta_2)d\langle\rho_2(\theta_2)A\rho_1(\theta_1)\xi, \eta\rangle\end{aligned}$$

This implies that $f(m,n)=\langle AU_1^m\xi, U_2^n\eta\rangle$, $-\infty < m, n < \infty$, is a Fourier-Stieltjes transform of some complex regular measure on $K\times K$. The proof is complete.

By Lemma 1 and [5, Theorem 2.7.2] it follows that the sequence $\langle\hat{\mu}(n,n)\rangle_{\infty=-\infty}^{n}$ is the sequence of Fourier-Stieltjes coefficients of some complex regular measure on the torus K.

Lemma 2. *Let G be a locally compact group and let $T(a)$ be a continuous affine transformation of G such that there is an element w in G such that $\{T(a)^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then T is bi-continuous.*

Proof. A locally compact group G which contains a countable dense set is σ-compact, and so T is an open automorphism by [3, Theorem (5.29)].

For x in G let $V(x)$ be the unitary operator on $L^2(,\lambda)$, where λ is a left invariant Haar measure on G, defined by

$$V(x)f(y)=f(xy) \qquad (y\in G, f\in L^2(G,\lambda)).$$

Let $T(a)$ be as in Lemma 2 then there is $\delta>0$ such that $\lambda(T(a)(E)) =\delta\lambda(E)$ for all Borel sets E of G. Then the operator $U(a)$ on $L^2(G,\lambda)$ defined by

$$U(a)f(y)=\delta f(T(a)y)=\delta f(a\cdot Ty) \qquad (y\in G, f\in L^2(G,\lambda))$$

is unitary and $U(a)^{-1}f(y)=\delta^{-1}f(T(a)^{-1}y)=\delta^{-1}f(T^{-1}a^{-1}\cdot T^{-1}y)$. We will denote by e the identity element in G and by U the unitary operator $U(e)$. Then we have the following

Lemma 3. *$V(T(a)^n(x))=U^{-n}V(x)U(a)^n$ for every integer n and for every x in G.*

Proof. Let x in G, n an integer, and $f\in L^2(G,\lambda)$. Then

$$\begin{aligned}&U^{-n}V(x)U(a)^n f(y)=U^{-n}V(x)[\delta^n f(T(a)^n(y)] \\ &\quad =U^{-n}[\delta^n f(T(a)^n(xy))]=f(T(a)^n(x\cdot T^{-n}y))=f(T(a)^n(x)\cdot y) \\ &\quad =V(T(a)^n(x))f(y).\end{aligned}$$

Theorem 1. *Let G be a locally compact totally disconnected non-discrete group and let $T(a)$ be a continuous affine transformation of G such that there is an element w in G such that $\{T(a)^n(w) \mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in G. Then G is compact.*

Proof. Let N be a compact open subgroup of G and let λ be normalized so that $\lambda(N)=1$. Let $U(a)$ and V be as above. For x in G and n an integer we define

$$a_n(x)=\langle V(x)U(a)^n\chi_N, U^n\chi_N\rangle=\langle U^{-n}V(x)U(a)^n\chi_N, \chi_N\rangle,$$

where χ_N is the indicator function of N. From Lemma 3 it follows

$$\begin{aligned} a_n(x)&=\langle V(T(a)^n(x))\chi_N, \chi_N\rangle \\ &=\int_G \chi_N(T(a)^n(x)\cdot y)\chi_N(y)d\lambda(y) \\ &=\begin{cases} 1 \text{ if } x\in T(a)^{-n}(N) \\ 0 \text{ if } x\notin T(a)^{-n}(N). \end{cases} \end{aligned} \tag{1}$$

Thus $a_n(T(a)(x))=\langle V(T(a)^{n+1}(x))\chi_N, \chi_N\rangle=a_{n+1}(x)$ for every integer n.

By Lemma 1 and the note below it, (1) implies that the sequence $\langle a_n(x)\rangle_{n=-\infty}^{\infty}$ is a sequence of Fourier-Stieltjes coefficients of some idempotent measure on the torus K, therefore $\langle a_n(x)\rangle_{n=-\infty}^{\infty}$ differs from a periodic sequence in at most finitely many places (see [2] or [5, 3.1.6]), from which the sequences $\{\langle a_n(x)\rangle \mid x\in G\}$ are countable. But the set $M(x)$ defined by

$$\begin{aligned} M(x)&=\{y\in G \mid \langle a_n(y)\rangle=\langle a_n(x)\rangle\} \\ &=\bigcap_{n=-\infty}^{\infty} T(a)^{-n}(N^{\varepsilon_n}), \end{aligned}$$

where $N^{\varepsilon_n}=N$ if $\varepsilon_n=a_n(x)=1$ and $N^{\varepsilon_n}=G\cap N^C$ if $\varepsilon_n=a_n(x)=0$, is an intersection of closed sets, and so it is closed. Thus the Baire category theorem implies that there is at least one element x in G such that $M(x)$ has non-void interior. Then the set

$$\begin{aligned} M^*(x)&=\bigcup_{j=-\infty}^{\infty} T(a)^j(M(x)) \\ &=\{y\in G \mid \langle a_n(y)\rangle=\langle a_{n+k}(x)\rangle \text{ for some integer } k\} \end{aligned}$$

must contain the set $\{T(a)^j(w) \mid j=0, \pm1, \pm2, \cdots\}$ for $M(x)$ is $T(a)$-invariant and $M(x)\subset M^*(x)$.

If $a_n(x)=0$ for all but finitely many n, let $k=1+\max\{|m-n| \mid a_m(x)=a_n(x)=1\}$. Since $\{T(a)^n(w) \mid n=0, \pm1, \pm2, \cdots\}$ is dense in G there are two integers m and n such that $|m-n|\geqq k$ and $T(a)^m(w)$, $T(a)^n(w)$ belong to N, whence $T(a)^{-m}(N)\cap T(a)^{-n}(N)$ is non-void open in G and disjoint from $M^*(x)$, which is impossible. Thus $a_n(x)=1$ for infinitely many n, from which and the essentially periodic property of the sequence $\langle a_n(x)\rangle$ it can be chosen a positive integer p such that in every interval of length p there is at least one n for which $a_n(x)=1$. This demonstrates

$$M^*(x)\subset N\cup T(a)(N)\cup\cdots\cup T(a)^p(N).$$

Thus the compactness of G follows. The proof is complete.

Corollary. *Let G be a locally compact totally disconnected non-discrete group and let a be an element in G such that $\{a^n \mid n=0, \pm1, \pm2, \cdots\}$ is dense in G. Then G is compact.*

Theorem 2. *Let G be a locally compact totally disconnected non-discrete group with a countable open basis and let $T(a)$ be a continuous affine transformation of G. If $T(a)$ is ergodic with respect to a Haar* *measure λ on G then G is compact.*

Proof. Let $\{O_j \mid j=1,2,3,\cdots\}$ be a countable open basis of G and put

$$E_j = G \cap \Big(\bigcup_{n=-\infty}^{\infty} T(a)^n(O_j)\Big)^C \quad \text{and} \quad E=\bigcup_j E_j.$$

Since $T(a)$ is ergodic, $\lambda(E_j)=0$ for all j, whence $\lambda(E)=0$. Therefore $T(a)$ has a dense orbit $\{T(a)^n(x) \mid n=0, \pm 1, \pm 2, \cdots\}$ for almost all x in G. Thus Theorem 1 implies that G is compact.

Theorem 3. *Let G be a locally compact totally disconnected (not necessarily non-discrete) group and let T be a continuous automorphism which has a dense orbit in G. Then G is compact.*

The proof is essentially identical with it of Theorem 1, and so it is sufficient to see that for every open subgroup N of G and for every pair (m, n) of integers $T^m(N) \cap T^n(N)$ is non-void open in G.

Remark 1. The non-discreteness of G in Theorems 1 and 2 is not omitted. For let G be the additive group of integers with discrete topology and let I be the identity transformation of G. Then the affine transformation $I(1)$ defined by $I(1)(n)=1+I(n)=1+n$ for n in G is ergodic with respect to a Haar measure on G and has a dense orbit $\{I(1)^n(1) \mid n=0, \pm 1, \pm 2, \cdots\}=G$. But G is trivially non-compact.

Remark 2. The non-discreteness and the second countability of G in Theorem 2 can be omitted if a continuous automorphism of G is ergodic (cf. [4]). This is similar to the relation between Theorems 1 and 3.

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