243. On Quasi-k-Spaces

By Yoshio TANAKA

(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1970)

0. **Introduction.** In this paper, we shall treat the case that the product space is a quasi-k-space. In Section 1, we give definitions and preliminaries. In Section 2, we shall prove the following theorems;

(a): A space X is locally countably compact if and only if $X\times Y$ is a quasi-k-space for every sequential space Y.

(b): Let X and Y be sequential spaces. Then $X\times Y$ is a sequential space if and only if it is a quasi-k-space.

(c): If X is a sequential q-space, and Y is a k-space and a q-space, then $X\times Y$ is a k-space and a q-space.

Finally, in Section 3, we consider the product space of uncountably many spaces.

We assume all spaces are regular and T_2, and all maps are continuous and onto. The weak topology in the sense of J. Dugundji [5], will be used throughout this paper.

1. **Definitions and preliminaries.** A space X is called a quasi-k-space (sequential space) if a subset F of X is closed whenever $F\cap C$ is closed in C for every countably compact (compact metric) subset C of X by J. Nagata [12] (S. P. Franklin [6]) respectively. Quasi-k-spaces (Sequential spaces) are precisely the quotients of M-spaces defined by K. Morita [11] (metric spaces) respectively. Of course, sequential spaces are k-spaces and k-spaces are quasi-k-spaces. But the converses do not hold. Indeed, the Stone-Cěch compactification of a normal and non-compact space is not sequential, and a countably compact space A_1 constructed by J. Novák [13] is not a k-space.

Lemma 1.1. *Let $f_i: X_i\to Y_i$ $(i=1,2)$ be quotient and X_1 be sequential. If $Y_1\times Y_2$ has the weak topology with respect to $\mathfrak{F}=\{Y_1\times C$; C is closed countably compact in $Y_2\}$, then $f_1\times f_2$ is quotient. Especially, when f_2 is closed, the closedness of the subset C is omitted.*

Proof. From the fact that $f_2|f_2^{-1}(C)$ is quotient for every closed subset C of Y_2, and that if $f_1\times f_2|(f_1\times f_2)^{-1}(F)$ is quotient for every $F\in\mathfrak{F}$, then $f_1\times f_2$ is quotient, we can assume Y_2 is countably compact. Moreover, sequential spaces are the quotients of locally compact metric spaces by S. P. Franklin [6; Corollary 1.13], we can also assume X_1 is locally compact metric. Now, $f_1\times f_2=(f_1\times i_{Y_2})\cdot(i_{X_1}\times f_2)$ and $i_{X_1}\times f_2$ is quotient by J. H. C. Whitehead [15; Lemma 4], and $f_1\times i_{Y_2}$ is quotient by E. Michael [9; Theorem 4.1], $f_1\times f_2$ is quotient.

Lemma 1.2. (a) *If X is a sequential space, then a countably compact subset of X is always closed.* (b) *Let X be either normal or countably paracompact. If C is a countably compact subset of X, then $\bar{C}$ is also countably compact.*

Proof. (a) is easily proved by the definition of a sequential space. (b) Let X be normal. Assume that $\bar{C}$ is not countably compact. Then there exists a discrete set $\{x_i; i \in n\}$ contained in $\bar{C}$. Hence we can choose a discrete collection $\{V_i; i \in N\}$ of open subsets with $x_i \in V_i$ by the normality of X. Since $x_i \in \bar{C}$, there exists a sequence $\{y_i; i \in N\}$ with $y_i \in V_i \cap C$. But C is countably compact, the sequence $\{y_i; i \in N\}$ has a cluster point. This is impossible. In case X is a countably paracompact space, from F. Ishikawa [7], (b) is proved straightforwards.

Remark. In completely regular spaces, (b) need not be true. Indeed, let $X=[0,\Omega]\times[0,\omega]-(\Omega,\omega)$, where ω is the first non-finite ordinal and Ω is the first uncountable ordinal, and let $A=[0,\Omega)\times[0,\omega]$. Then A is countably compact but $\bar{A}=X$ is not countably compact.

According to E. Michael [8], a space X is called a q-space if each point of X has a sequence $\{U_i; i \in N\}$ of open nbds (=neighborhoods) such that $\bar{U}_{i+1}\subset U_i$, and if $x_i \in U_i$, then the sequence $\{x_i; i \in N\}$ has a cluster point. Such a sequence is called a q-sequence of nbds. Locally countably compact spaces, M-spaces, and spaces of pointwise countable type defined by A. Arhangel'skii [1] are all q-spaces. A q-space is a quasi-k-space by J. Nagata [12].

Lemma 1.3. *if* (a) *$X\times Y$ is a q-space, or* (b) *$X\times Y$ is a quasi-k-space and Y is normal or countably paracompact, or sequential, then $X\times Y$ has the weak topology with respect to $\mathcal{C}=\{X\times C; C$ is closed countably compact in $X\}$.*

Proof. (a): Put $X_1=X$ and $X_2=Y$. Let F be a subset of $X_1\times X_2$ such that $F\cap C$ is closed for every $C\in\mathcal{C}$. Let $(x_1,x_2)\in\bar{F}$ be given and V_i be any open nbd of x_i. Since $X_1\times X_2$ is a q-space, it has a q-sequence $\{W_{1j}\times W_{2j}; j\in N\}$ of a point (x_1,x_2). Let W'_{ij} be an open nbd of x_i such that $\bar{W}'_{ij+1}\subset W'_{ij}\subset V_i\cap W_{ij}$ for each $j\in N$. Since $(x_1,x_2)\in\bar{F}$, there exists a sequence $\{(x_{1j},x_{2j}); j\in N\}$ with $(x_{1j},x_{2j})\in F\cap(W'_{1j}\times W'_{2j})$. Put $C_2=\bigcap_{j=1}^{\infty}W_{2j}$ and $C'_2=\{x_{2j}; j\in N\}\cup\bigcap_{j=1}^{\infty}W'_{2j}$, then C_2 and C'_2 are closed countably compact in X_2, and the sequence $\{x_{1j},x_{2j}); j\in N\}$, whose closure is contained in a closed subset $F\cap(X_1\times C'_2)$, has a cluster point in $(V_1\times V_2)\cap(X_1\times C_2)$. Hence $(x_1,x_2)\in F$ by the closedness of $F\cap(X_1\times C_2)$. Therefore $X_1\times X_2$ has the weak topology with respect to $\mathcal{C}$. (b): Since every subset K of $X\times Y$ is contained in $X\times\overline{P_Y(K)}$, where P_Y is the projection of $X\times Y$ onto Y, and $\overline{P_Y(K)}$ is closed countably compact for every countably compact subset K of $X\times Y$ by Lemma 1.2,

$X\times Y$ has the weak topology with respect to $\mathcal{C}$.

From T. Chiba [4; Theorem 4], Lemma 1.1, and Lemma 1.3, we have

Proposition 1.4. *Let $f_i: X_i\to Y_i$ $(i=1,2)$ be quotient maps and X_1 be a sequential space. If* (a) *Y_1 and Y_2 are q-spaces, or* (b) *$Y_1\times Y_2$ is a quasi-k-space and Y_2 is either normal or countably paracompact, then $f_1\times f_2$ is quotient.*

Remark. Lemma 1.1 remains true if we replace the words "sequential" and "closed countably compact" by "a k-space" and "compact" respectively, which leads to E. Michael [10; Theorem 1.5].

2. Proof of theorems.

Theorem 2.1. *The following properties of a space X are equivalent.*

(a) *X is a locally countably compact space.*

(b) *$X\times Y$ is a quasi-k-space for every sequential space Y.*

(c) *$X\times Y$ is a quasi-k-space for every paracompact sequential space Y.*

Proof. (a)→(b): Let Y_0 be the topological sum of the family $\{C;$ C is compact metric in $Y\}$, and f be the quotient map of Y_0 onto Y. Then $f\times i_X$ is quotient by E. Michael [9; Theorem 4.1 (a)→(b)]. Since $Y_0\times X$ is a quasi-k-space and quotients of quasi-k-spaces are quasi-k-spaces, $X\times Y$ is a quasi-k-space.

(b)→(c): Obvious.

(c)→(a): Assume that X is not countably compact.
From E. Michael [9; Theorem 4.1 (c)→(a)], there exists a closed map g of a metric space onto a space Y such that $i_X\times g$ is not quotient. Since Y is a paracompact sequential space, $X\times Y$ is a quasi-k-space by the hypothesis. From Lemma 1.1 and the proof of Lemma 1.3 (b), $i_X\times g$ is quotient, which is impossible.

Theorem 2.2. *Let X and Y be sequential spaces. Then $X\times Y$ is sequential if and only if it is a quasi-k-space.*

Proof. The "only if" part is obvious.

"if": Let X_0 and Y_0 be the topological sums of the families $\{C; C$ is compact metric in $X\}$ and $\{C'; C'$ is compact metric in $Y\}$ respectively, and let $f: X_0\to X$, $g: Y_0\to Y$ be quotient. Since $X\times Y$ is a quasi-k-space and Y is sequential, from Lemma 1.1 and Lemma 1.3 (b), $f\times g$ is quotient. Since $X_0\times Y_0$ is metric, $X\times Y$ is sequential.

From T. Chiba [4; Theorem 4] and J. Nagata [12; Corollary to Theorem 1], and Theorem 2.2, we have

Corollary 2.3. *If X and Y are sequential q-spaces, then $X\times Y$ is a sequential q-space.*

From Theorem 2.1 and Theorem 2.2, we have

Corollary 2.4 (T. K. Boehme [3]). *If X is a locally countably compact, sequential space and Y is a sequential space, then $X \times Y$ is a sequential space.*

According to P. Bacon [2], a space X is called isocompact if every closed countably compact subset of X is compact. Paracompact spaces, σ-spaces, and developable spaces are all isocompact spaces.

Theorem 2.5. *Let X be either normal or countably paracompact. If* (a) *X is isocompact and Y is a k-space, or* (b) *X is a k-space and Y is sequential, then $X \times Y$ is a k-space if and only if it is a quasi-k-space.*

Proof. Let $X \times Y$ be a quasi-k-spaces. (a) $X \times Y$ has the weak topology with respect to $\{\overline{P_X(K)} \times Y\,;\, K \text{ is countably compact in } X \times Y\}$, where P_X is the projection of $X \times Y$ onto X, and $\overline{P_X(K)}$ is compact for every countably compact subset K of $X \times Y$ by Lemma 1.2 (b). From Remark to Proposition 1.4, $X \times Y$ is a k-space by the same way as in the proof of Theorem 2.2. (b) Similarly, from Proposition 1.4 (b), $X \times Y$ is a k-space.

Remark. Lemma 1.3, Proposition 1.4, and Theorem 2.5 remain true if we replace the words "normal" and "countably paracompact" by "locally normal" and "locally countably paracompact" respectively.

According to A. Arhangel'skii [1], a space X is called of pointwise countable type if each point of X is countained in a compact subset having a countable local basis.

Theorem 2.6. *Let X be a k-space and a q-space. If Y is either a space of pointwise countable type or a sequential q-space, then $X \times Y$ is a k-space and a q-space.*

Proof. From T. Chiba [4; Theorem 4], $X \times Y$ is a q-space. If Y is of pointwise countable type, put $X = X_1$ an $Y = X_2$, then the subsets C_2, C_2' in the proof of Lemma 1.3 (a) are compact. Hence $X \times Y$ has the weak topology with respect to $\{X \times C\,;\, C \text{ is compact in } Y\}$. Therefore, by the same way as in the proof of Theorem 2.5 (a), $X \times Y$ is a k-space. Similarly, if Y is a sequential q-space, $X \times Y$ is a k-space by Proposition 1.4 (a).

3. In this section, we consider the product space of uncountably many spaces.

From the fact that every subset K of $\prod_{\alpha \in A} X_\alpha$ is contained in $\prod_{\alpha \in A} \overline{P_\alpha(K)}$, where P_α is the projection of $\prod_{\alpha \in A} X_\alpha$ onto X_α, and Lemma 1.2, we have

Theorem 3.1. *Let X_α be an isocompact space which is normal, or countably paracompact, or sequentia l for each $\alpha \in A$. Then $\prod_{\alpha \in A} X_\alpha$ is isocompact, and it is a k-space if and only if it is a quasi-k-space.*

Lemma 3.2. *If $\prod_{\alpha \in A} X_\alpha$ is a q-space, then all but a countable number of spaces X_α must be countably compact.*

Proof. Assume that there exists an uncountable subset A' of A such that $X_{\alpha'}$ is not countably compact for each $\alpha' \in A'$. Then $X_{\alpha'}$ contains a copy of N, say $N_{\alpha'}$, as a closed subset for each $\alpha' \in A'$. Since $\prod_{\alpha' \in A'} N_{\alpha'}$ is closed in $\prod_{\alpha \in A} X_\alpha$, it is a q-space. Then there exists a q-sequence $\mathcal{U} = \{U_i ; i \in N\}$ of a point of $\prod_{\alpha' \in A'} N_{\alpha'}$, such that each U_i is an open basic subset of $\prod_{\alpha' \in A'} N_{\alpha'}$. Put $K = \bigcap_{i=1}^{\infty} U_i, K$ is countably compact and for any oped subset 0 of $\prod_{\alpha' \in A'} N_{\alpha'}$ containing K we can find U_i satisfying $K \subset U_i \subset 0$. Let $B = \{\alpha' ; \alpha' \in A', P_{\alpha'}(U) \neq N_{\alpha'}$ for some U in $\mathcal{U}\}$, where $P_{\alpha'}$ is the projection of $\prod_{\alpha' \in A'} N_{\alpha'}$ onto $N_{\alpha'}$. Then there exists an element α'_0 in $A' - B$. Since $P_{\alpha'_0}(K)$ is compact, $P_{\alpha'_0}(K) \neq N_{\alpha'_0}$. Put $0 = P_{\alpha'_0}(K) \times \prod_{\alpha' \neq \alpha'_0} N_{\alpha'}$, 0 is an open subset of $\prod_{\alpha \in A'} N'_\alpha$ containing K. But $U_i \not\subset 0$ for each $i \in N$, which is impossible.

Theorem 3.3. *Let X_α be an isocompact space for each $\alpha \in A$. Then $\prod_{\alpha \in A} X_\alpha$ is a q-space if and only if it is of pointwise countable type. Especially, if each X_α is a paracompact M-space, then the following properties of the product space $\prod_{\alpha \in A} X_\alpha$ are equivalent.*

(a) *a q-space.*, (b) *an M-space.*, (c) *a paracompact space.*

Proof. Since spaces of pointwise countable type are q-spaces, the "if" part is proved. The "only if" part follows from the fact that isocompact q-spaces are of pointwise countable type, and A. Arhangel'skii [1; Theorem 3.9′], and Lemma 3.2.

Since M-spaces are q-spaces, from K. Morita [11; Theorem 6.4] and Lemma 3.2, (a)↔(b) and (b)→(c) are proved. (c)→(b) follows from A. H. Stone [14; Corollary to Theorem 4] and K. Morita [11; Theorem 6.4].

References

[1] A. Arhangel'skii: Bicompact sets and the topology of spaces. Translation of the Moskow Math. Soc., 1–62 (1965).

[2] P. Bacon: The compactness of countably compact spaces. Pacific. J. Math., **32**, 587–592 (1970).

[3] T. K. Boehme: Linear s-spaces. Proc. Symp. Convergence Structures. U. of Oklahoma (1965).

[4] T. Chiba: On q-spaces. Proc. Japan Acad., **45**, 453–456 (1969).

[5] J. Dugundji: Topology. Allyn and Bacon Inc. Boston (1967).

[6] S. P. Franklin: Spaces in which sequences suffice. Fund. Math., **57**, 102–114 (1967).

[7] F. Ishikawa: On countably paracompact spaces. Proc. Japan Acad., **31**, 686–687 (1955).

[8] E. Michael: A note on closed maps and compact sets. Israel. J. Math., **2**, 173–176 (1964).

[9] ——: Local compactness and cartesian products of quotient maps and k-spaces. Ann. Inst. Fourier, Grenoble., **18** (2), 281–286 (1968).

[10] ——: Bi-quotient maps and cartesian products of quotient maps. Ann. Inst. Fourier, Grenoble., **18** (2), 287–302 (1968).

[11] K. Morita: Products of normal spaces with metric spaces. Math. Annalen., **154**, 365–382 (1964).

[12] J. Nagata: Quotient and bi-quotient spaces of M-spaces. Proc. Japan Acad., **45**, 25–29 (1969).

[13] J. Novák: On the cartesian product of two compact spaces. Fund. Math., **40**, 106–112 (1953).

[14] A. H. Stone: Paracompactness and product spaces. Bull. Amer. Math. Soc., **54**, 977–982 (1948).

[15] J. H. C. Whitehead: A note on a theorem of Borsuk. Bull Amer. Math. Soc., **54**, 1125–1132 (1958).