159. *Products of M-Spaces*

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The spaces considered here are always completely regular T_1-spaces and mappings are continuous. We have showed in the previous paper [4] that the product of M-spaces need not be an M-space. In this paper, introducing a new class $\mathfrak{C}(M)$ of M-spaces, we shall prove in § 2 the following main theorem:

$X \in \mathfrak{C}(M)$ if and only if the product $X \times Y$ is an M-space for every M-space Y.

In § 3, we shall show that $\mathfrak{C}(M)$ contains the class $\mathfrak{C}(*)$ which contains all M-spaces X such that X satisfies one of the following conditions: (a) X satisfies the first axiom of countability, (b) X is locally compact and (c) X is paracompact (see [2], p. 897), moreover $\mathfrak{C}(M)$ contains the class $\mathfrak{C}(C)$ (§ 1 below).

§ 1. Definitions and preliminaries. A space X is called an *M-space*, the notion of which is introduced by K. Morita [6], if there exists a normal sequence $\{\mathcal{U}_i\}$ of open coverings of X satisfying the following condition (M):

(M) If $\{K_i\}$ is a sequence of non-empty closed subsets of X such that $K_{i+1} \subset K_i$ and $K_i \subset \mathrm{St}(x_0, \mathcal{U}_i)$ for each i and some fixed point $x_0 \in X$, then $\cap K_i \neq \emptyset$.

In the following we call $\{\mathcal{U}_i\}$ mentioned above, for simplicity, an *M-normal sequence* of X. A sequence $\{x_i\}$ in X is said to be an *(M)-sequence* if $x_i \in \mathrm{St}(x_0, \mathcal{U}_i)$ for every i and some fixed point x_0 of X and some M-normal sequence $\{\mathcal{U}_i\}$ of X. In [2] the class $\mathfrak{C}(*)$ has been introduced as the set of all M-spaces satisfying the following condition (*):

(*) Any (M)-sequence has a subsequence whose closure is compact.

The symbol $\mathfrak{C}(C)$ denotes the class of all spaces P such that the product $P \times Q$ is countably compact for every countably compact space Q. This class has been introduced by Frolík [1] and it is obvious that $P \in \mathfrak{C}(C)$ implies that $F \in \mathfrak{C}(C)$ for every closed subset F of P. We shall consider the class $\mathfrak{C}(M)$ consisting of all M-spaces X satisfying the following condition (CM):

(CM) For any discrete subsequence N of any (M)-sequence of X and for any non-empty subset S of $K-X$ where K is any compactification of X, the subspace $N \cup S$ of K is not countably compact.

For a mapping φ from X onto Y, we denote by Φ the Stone-extension of φ from βX onto βY.

1.1 ([1], Theorem 3.3). *$P \notin \mathfrak{C}(C)$ if and only if P satisfies the following condition: There exists an infinite discrete subset N of P such that for every compactification K of P there exists a subset S of $K-P$ such that the subspace $N \cup S$ of K is countably compact.*

1.2 ([1], 3.9). *The product of a countable subfamily of $\mathfrak{C}(C)$ belongs to $\mathfrak{C}(C)$.*

From 1.1 we have

1.3 ([5], Theorem 1.6). *The following conditions are equivalent.*

1) $X \in \mathfrak{C}(C)$.

2) *For every infinite discrete subset N of X, there is a compactification K such that for any subset S of $K-X$, the subspace $N \cup S$ of K is not countably compact.*

3) *For every infinite discrete subset N of X, the subspace $N \cup S$ of K is not countably compact where K is any compactification of X and S is any subset of $K-X$.*

1.4. Similarly to 1.3), we can replace the phrase "K is any compactification of X" by the phrase "K is a compactification of X" in the condition (CM).

1.5 ([2], Lemma 2.1). *Let $\{\mathcal{U}_i\}$ and $[\mathcal{V}_i\}$ be normal sequences of X and Y respectively. Then $\{\mathcal{W}_i ; \mathcal{W}_i=\{U\times V ; U \in \mathcal{U}_i, V \in \mathcal{V}_i\}\}$ is a normal sequence of $X \times Y$.*

1.6 ([2], Lemma 2.2). *For each n, let $\{\mathcal{U}(n, i) ; i=1, 2, \cdots\}$ be a normal sequence of X_n. Then $\{\mathcal{U}_i ; \mathcal{U}_i=\{U_1 \times \cdots \times U_i \times \boldsymbol{P}_{n>i} X_n ; U_j \in \mathcal{U}(j, i), j=1,2, \cdots, i\}\}$ is a normal sequence of $\boldsymbol{P} X_n$.*

§2. Proof of Main Theorem. *Necessity.* Let $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ be M-sequences of X and Y respectively. $\{\mathcal{W}_i\}$ constructed in (1.5) is normal. We shall show that $\{\mathcal{W}_i\}$ satisfies the condition (M), or equivalently, the condition (M_0): if $\{(x_i, y_i)\}$ is an (M)-sequence with respect to $\{\mathcal{W}_i\}$, then $\{(x_i, y_i)\}$ has an accumulation point (see [2], p. 897). By the methods of construction of $\{\mathcal{W}_i\}$, it is obvious that $\{x_i\}$ and $\{y_i\}$ are (M)-sequences of X and Y respectively. Since every infinite Hausdorff space contains an infinite discrete subset, there exists a discrete subsequence N of $\{x_i\}$. $\mathrm{cl}_{\beta X}N$ is a compactification of N. $X \in \mathfrak{C}(M)$ implies that $N \cup S$ is not countably compact for every subset S of $\beta X - X$. If we restrict S to $\mathrm{cl}_{\beta X}N - N$, then according to (1.3) we have $\mathrm{cl}_X N \in \mathfrak{C}(C)$ (notice that the closure of an (M)-sequence in an M-space is countably compact). Let $L=\{y_i ; i=1, 2, \cdots\}$. Since Y is an M-space and $\{y_i\}$ is an (M)-sequence, $\mathrm{cl}_Y L$ is countably compact. Thus $\mathrm{cl}_X N \times \mathrm{cl}_Y L$ is also countably compact and hence $\{(x_i, y_i)\}$ has an accumulation point, which leads to the fact that $X \times Y$ is an M-space.

Sufficiency. Now suppose that there exists an (M)-sequence of an M-space X whose some discrete subsequence $N=\{x_i\}$ has the property such that the subspace $N\cup S$ of K is countably compact where K is a compactification of X and S is some subset of $K-X$. Without loss of generality we can assume that S is contained in $\mathrm{cl}_K N-N$. We shall show that there is a countably compact space Y such that the product $Y\times Y$ is not an M-space (notice that every countably compact space is an M-space). X being an M-space, $\mathrm{cl}_X N$ is countably compact and hence there exists a point x^* in $\mathrm{cl}_X N-N$. Let us put $Y=N\cup S\cup\{x^*\}$. Y is obviously countably compact. From the assumption that $X\times Y$ is an M-space and hence $\mathrm{cl}_X N\times Y$ is also an M-space, we shall deduce a contradiction. Let $\{\mathcal{W}_i\}$ be an M-normal sequence of $L\times Y$ where $L=\mathrm{cl}_X N$. Since $x^*\in L-N$, we have a subsequence $\{x_{n_i}\}$ of N such that

$$(x_{n_i}, x_{n_i})\in \mathrm{St}((x^*, x^*), \mathcal{W}_i)$$

for each i (change indices if necessary). Let U_i be an open neighborhood (in K) of x^* such that for each i

$$\mathrm{cl}_k U_i\cap\{x_1, x_2, \cdots, x_i\}=\varnothing$$

and

$$(\mathrm{cl}_K U_i\times \mathrm{cl}_K U_i)\cap(L\times Y)\subset \mathrm{St}((x^*, x^*), \mathcal{W}_i).$$

On the other hand, since $\{\mathrm{cl}_K U_i\cap(N\cup S)\}$ has a finite intersection property and $N\cup S$ is countably compact, its total intersection contains a point s^*. By the method of construction of $\{U_i\}$ it is easily seen that $s^*\in S$. $s^*\neq x^*$ implies that there exists an open neighborhood V (in K) of s^* such that $x^*\notin \mathrm{cl}_K V$. Let us put

$$K_i=\mathrm{cl}_{L\times Y}((U_i\times U_i)\cap(V\times V))\cap \Delta(N)$$

where $\Delta(N)$ is the diagonal of $N\times N$. Obviously $K_i\subset \mathrm{St}((x^*, x^*), \mathcal{W}_i)$ for each i and hence we have $\cap K_i\neq\varnothing$ by the condition (M) because $L\times Y$ is an M-space. On the other hand, $\mathrm{cl}_K U_i\cap\{x_1, x_2, \cdots, x_i\}=\varnothing$ implies that $K_i\cap(\{x_1, x_2, \cdots, x_i\}\times\{x_1, x_2, \cdots, \cdots, x_i\})=\varnothing$ and hence $(\cap K_i)\cap(N\times N)=\varnothing$. Moreover $x^*\times \mathrm{cl}_K V$ leads to $(\cap K_i)\cap(L\times Y)=\varnothing$, i.e., $\cap K_i=\varnothing$, which is a contradiction.

§3. Properties of the class $\mathfrak{C}(\mathrm{M})$. 3.1. $\mathfrak{C}(C)\subset\mathfrak{C}(M)$. Let $X\in\mathfrak{C}(C)$. Since X is countably compact, we can take as an M-normal sequence $\{\mathcal{U}_i\}$ a sequence of coverings each of which consists of only one element X and hence any discrete sequence N is an (M)-sequence with respect to $\{\mathcal{U}_i\}$, which shows that $X\in\mathfrak{C}(M)$.

3.2. $\mathfrak{C}(*)\subset\mathfrak{C}(M)$. Let $X\in\mathfrak{C}(*)$. For a discrete subsequence of an (M)-sequence, there exists a subsequence N whose closure is compact by the condition $(*)$. This implies that for any compactification K of X and any subset S of $K-X$, the subspace $N\cup S$ of K is not countably compact. This shows that $X\in\mathfrak{C}(M)$.

From the proof of sufficiency and the fact that countably compact spaces are M-space we have the following

3.3. Corollary. *$X \in \mathfrak{C}(M)$ if and only if the product $X \times Y$ is an M-space for every countably compact space Y.*

Let $X_n \in \mathfrak{C}(M)$ and $X = \boldsymbol{P}X_n$ and $\{x(i)\}$ be a discrete sequence of X. Without loss of generality, we can select a suitable subsequence $\{x(i_n)\}$ of $\{x(i)\}$ whose projection $\{x_j(i_n)\}$ on X_j is discrete. By this fact and (1.2) and (1.5) we have

3.4. *If $X_n \in \mathfrak{C}(M)$ $(n=1, 2, \cdots)$, then $\boldsymbol{P}X_n$ is an M-space.*

3.5. *If φ is a perfect mapping from X onto Y and $Y \in \mathfrak{C}(M)$, then $X \in \mathfrak{C}(M)$.*

Let Z be any M-space. The mapping $\psi: X \times Z \to Y \times Z$ defined by $\psi(x, z) = (\varphi(x), z)$ is perfect and hence $X \times Z$ is an M-space.

3.6. *Let φ be a quasi-perfect mapping from X onto Y and $X \in \mathfrak{C}(M)$. If X or Y is normal, then $Y \in \mathfrak{C}(M)$.*

From ([3], [7]), Y is an M-space. Let N be a discrete subsequence of an (M)-sequence of Y with respect to an M-normal sequence $\{\mathcal{U}_i\}$ such that the subspace $N \cup S$ is countably compact for some subset S of $\beta Y - Y$. $\Phi^{-1}(N) \cup \Phi^{-1}(S)$ is countably compact. Let x_y be a point of $\varphi^{-1}(y)$ and put $N_1 = \{x_y ; y \in N\}$. It is obvious that N_1 is a discrete (M)-sequence with respect to an (M)-sequence $\{\varphi^{-1}(\mathcal{U}_i)\}$ and $N_1 \cup \Phi^{-1}(S)$ is countably compact, which completes the proof.

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