# 147. Korteweg-deVries Equation. III

## Global Existence of Asymptotically Periodic Solutions

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**1. Introduction.** In the preceding note [1] we show the global existence of the smooth solutions of the Cauchy problem for the $KdV$ equation. That is for any initial data $f(x) \in \mathcal{E}^{\infty}_{L^2}(R^1) = \mathcal{E}^{\infty}_{L^2}$ and for any inhomogeneous term $g(x, t) \in \mathcal{E}^{\infty}_{t}(\mathcal{E}^{\infty}_{L^2})$ the Cauchy problem for the $KdV$ equation

$$\begin{cases} D_t u + uDu + D^3 u + g(x, t) = 0 & (x, t) \in R^1 \times [0, \infty) \\ u(x, 0) = f(x) \quad x \in R^1 & \left(D_t = \dfrac{\partial}{\partial t}, \quad D = \dfrac{\partial}{\partial x}\right) \end{cases}$$

has uniquely the global solution $u(x, t) \in \mathcal{E}^{\infty}_{t}(\mathcal{E}^{\infty}_{L^2})$ $(0 \leq t < \infty)$. Moreover we may replace the functional space $\mathcal{E}^{\infty}_{L^2}$ by the functional space $\mathcal{P}^{\infty}_{l}$ (see Definition 1).

In this note we extend slightly the preceding results [1] and show the global existence of the asymptotically periodic solutions of the Cauchy problem for the $KdV$ equation.

Detailed proof will be published elsewhere.

**2. Global existence theorems.**

**Definition 1.** $f(x)$ belongs to the functional space $\mathcal{P}^{k}_{l}$ if and only if $f(x)$ is a periodic function with period $l$, and belongs to $\mathcal{E}^{k}_{L^2 \mathrm{loc}}(R^1)$ ($k$ is a non negative integer or $\infty$).

**Definition 2.** $f(x)$ is called asymptotically periodic if and only if $f(x)$ belongs to the functional space $Q^{k}_{l} = \mathcal{P}^{k}_{l} + \mathcal{E}^{k}_{L^2}$ for some $k$ and $l$. Here $+$ signe represents the direct sum of the two Hilbert spaces $\mathcal{P}^{k}_{l}$ and $\mathcal{E}^{k}_{L^2}$.

Consider the Cauchy problem for the $KdV$ equation (with dissipative lower order terms)

$$(1) \quad \begin{cases} D_t u + uDu + D^3 u - \mu D^2 u + a(x, t)Du + b(x, t)u + g(x, t) = 0 \\ \qquad\qquad\qquad\qquad\qquad (x, t) \in R^1 \times [0, \infty) \\ u(x, 0) = f(x) \qquad\qquad x \in R^1 \end{cases}$$

**Assumption 1.** $\mu \geq 0.$ $\quad a(x, t), b(x, t) \in \mathcal{E}^{\infty}_{t}(\mathcal{P}^{\infty}_{l})$

**Main theorem.** We assume Assumption 1. For any initial data $f(x) = f_0(x) + f_1(x) \in Q^{\infty}_{l}$ and for any inhomogeneous term $g(x, t) = g_0(x, t) + g_1(x, t) \in \mathcal{E}^{\infty}_{t}(Q^{\infty}_{l})$ the Cauchy problem for the $KdV$ equation (1) has uniquely the global solution $u(x, t) \in \mathcal{E}^{\infty}_{t}(Q^{\infty}_{l})$ $(0 \leq t < \infty)$. Moreover

$u(x, t)$ decomposes into the sum of the periodic part $u_0(x, t) \in \mathcal{E}_t^\infty(\mathcal{P}_l^\infty)$ and the decaying part $u_1(x, t) \in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty)$. $u_0(x, t)$ and $u_1(x, t)$ are the solutions of the Cauchy problems (2) and (3) respectively.

$$(2)\quad \begin{cases} D_t u_0 + u_0 D u_0 + D^3 u_0 - \mu D^2 u_0 + a(x, t) D u_0 + b(x, t) u_0 + g_0(x, t) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (x, t) \in R^1 \times [0, \infty) \\ u_0(x, 0) = f_0(x) \qquad\qquad x \in R^1 \end{cases}$$

$$(3)\quad \begin{cases} D_t u_1 + u_1 D u_1 + D^3 u_1 - \mu D^2 u_1 + D(u_0 u_1) + a(x, t) D u_1 + b(x, t) u_1 \\ \quad + g_1(x, t) = 0 \qquad\qquad (x, t) \in R^1 \times [0, \infty) \\ u_1(x, 0) = f_1(x) \qquad\qquad x \in R^1. \end{cases}$$

This follows easily from the following two global existence theorems.

**Assumption 2.** $\mu \geq 0$. $a(x, t), b(x, t) \in \mathcal{E}_t^\infty(\mathcal{B}^\infty)$

**Theorem 1.** *We assume Assumption 2. For any initial data $f(x) \in \mathcal{E}_{L^2}^{3(k+1)}$ and for any inhomogeneous term $g(x, t) \in \mathcal{E}_t^{k+1}(L^2) \cap [\mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})]$ the Cauchy problem for the KdV equation* (1) *has uniquely the global solution $u(x, t) \in \mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})$.*

Using Sobolev's lemma we obtain

**Corollary 1.** *If*

$$f(x) \in \mathcal{E}_{L^2}^{3(k+3)}$$
$$g(x, t) \in \mathcal{E}_t^{k+3}(L^2) \cap [\mathcal{E}_t^{k+2}(L^2) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3(k+2)})]$$

*then for any $i, j$ such that $i + j \leq k$*

$$D_t^i D^{3j} u(x, t) \in \mathcal{B}^0(R^1 \times [0, T]) \quad \text{for} \quad \forall T > 0.$$

**Theorem 2.** *In the statement of Theorem* 1 *we can replace the functional space $\mathcal{E}_{L^2}^k$ by the functional space $\mathcal{P}_l^k$. But in this case we must replace Assumption* 2 *by Assumption* 1.

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**3. Proof of the main theorem.** We only sketch the outline of the proof. It suffices to prove Theorem 1, for the proof of Theorem 2 goes analgously.

To prove Theorem 1 we need the a priori estimate (Theorem 4) and the local existence theorem (Theorem 5).

To obtain the a priori estimate we use the results of Miura-Gardner-Kruskal [2]. We state their results in a slightly modified form.

**Theorem 3.** *For any non negative integer $k$, the KdV equation $D_t u + u D u + D^3 u = 0$ has the polynomial conserved density of the form*

$$T_k(u) = (D^k u)^2 + c_k u (D^{k-1} u)^2 + Q_k(u, \cdots, D^{k-2} u)$$
$$T_0(u) = u^2.$$

Here $c_k$ is a constant independent of $u$, $Q_k$ is a polynomial of rank $k+2$.

**Definition 3.** $T(u)$ is called polynomial conserved density if and only if $T$ is a polynomial in finite number of $D^k u$'s ($k=0, 1, 2, \cdots$) and there exists $X$ which is also polynomial in $D^k u$'s such that $D_t T = DX$.

**Definition 4.** A polynomial $Q$ is called rank $m$ if $Q$ is a sum of finite number of monomials of same rank $m$. For a monomial we define

$$\text{rank}\,[u^{\alpha_0}(Du)^{\alpha_1}\cdots(D^l u)^{\alpha_l}] = \sum_{j=0}^{l} \frac{1}{2}(j+2)\alpha_j$$

By integrating these polynomial conserved densities on the whole $x$-axis we get step by step following infinite number of a priori estimate for the solutions of the $KdV$ equation (1).

**Theorem 4.** *For any non negative integer $k$, the solutions of the $KdV$ equation* (1) *satisfy a priori estimate of the form*

$$|||D^k u|||_t \le U_k(t, |a|_t, \cdots, |D^k a|_t, |b|_t, \cdots, |D^k b|_t, \\ ||f||, \cdots, ||D^k f||, |||g|||_t, \cdots, |||D^k g|||_t).$$

Here

$$|||u|||_t = \sup_{0\le s\le t} ||u(s)||, \qquad ||u|| = ||u||_{L^2(R^1)}$$

$$|a|_t = \sup_{0\le s\le t} |a(s)|, \qquad |a| = |a|_{\mathcal{B}^0}.$$

$U_k$ are positive valued smooth monotone increasing function in each arguments. $U_0$ contains $|Da|_t$ exceptionally.

**Remark.** $U_k$ are independent of $\mu$ such that $0 \le \mu \le \mu_0$ (for some $\mu_0 > 0$ fixed).

The local existence theorem is obtained by the method of successive approximation

$$(4) \qquad u_0(x, t) = f(x) \qquad (x, t) \in R^1 \times [0, \infty)$$

$$(5) \quad \begin{cases} D_t u_n + u_{n-1} D u_n + D^3 u_n - \mu D^2 u_n + a(x, t) D u_n + b(x, t) u_n \\ \quad + g(x, t) = 0 \qquad\qquad (x, t) \in R^1 \times [0, \infty) \\ u_n(x, 0) = f(x) \qquad\qquad x \in R^1 \qquad n = 1, 2, 3, \cdots. \end{cases}$$

By induction in $n$ and $k$ we obtain following uniform (with respect to $n$) local energy estimate for approximate solutions $u_n$.

**Proposition 1.** *For any non negative integer $k$, if we take*

$$t_k = \min\left\{1, \frac{\log M}{C_1}, \frac{1}{C_k}\right\}$$

$$t_0 = \min\left\{1, \frac{\log M}{C_1}\right\} \quad \textit{exceptionally}$$

*then for any $i, j$ such that $i + j \le k$*

$$|||D_t^i D^{3j} u_n|||_{t_k} \le c_{i,j} \qquad n = 0, 1, 2, \cdots.$$

Here we use the following notations

$$u(0) = f(x)$$

$$u^{(1)}(0) = -[u(0)Du(0) + D^3 u(0) - \mu D^2 u(0) + a(x, 0)Du(0) \\ + b(x, 0)u(0) + g(x, 0)]$$

$$u^{(k)}(0) = -\Big[\sum_{l=0}^{k-1} \binom{k-1}{l} u^{(k-1-l)}(0) D u^{(l)}(0) + D^3 u^{(k-1)}(0)$$
$$- \mu D^2 u^{(k-1)}(0) + \sum_{l=0}^{k-1} \binom{k-1}{l} D_t^{k-1-l} a(x,0) D u^{(l)}(0)$$
$$+ \sum_{l=0}^{k-1} \binom{k-1}{l} D_t^{k-1-l} b(x,0) u^{(l)}(0) + D_t^{k-1} g(x,0)\Big]$$
$$c_{k,0}^2 = M[\|u^{(k)}(0)\|^2 + |||D_t^k g|||_1^2 + \cdots + |||g|||_1^2 + 1]$$

$M$ is a fixed constant such that $M>1$.

$$c_{k-l-1,l+1} = c_{k-l,l} + C_{k-1} + |||D_t^{k-l-1} D^{3l} g|||_1$$
$$l = 0, 1, 2, \cdots, k-1$$

$C_k$ is a polynomial of $c_{i,j}$, $|D_t^i D^{3j} a|_1$, $|D_t^i D^{3j} b|_1$ $(i+j \le k)$ with positive coefficients.

From (5) we derive the following equations for the differences $u_{n+1} - u_n = \varphi_n$

$$(6)\quad \begin{cases} D_t\varphi_n + u_n D\varphi_n + \varphi_{n-1} D u_n + D^3\varphi_n - \mu D^2 \varphi_n + a(x,t) D\varphi_n \\ \quad + b(x,t)\varphi_n = 0 & (x,t) \in R^1 \times [0,\infty) \\ \varphi_n(x,0) = 0 & x \in R^1 \end{cases}$$

Using uniform estimate for $u_n$ in Proposition 1 we obtain the following estimate for $\varphi_n$

**Proposition 2.** *For any non negative integer $k$, if we take*

$$T_k = \min\left\{1, \frac{\log M}{C_{k+1}}, \frac{\rho}{(k+1)M}\right\} \quad 0<\rho<1 \quad \textit{fixed}$$

*then we have*

$$|||D_t^k \varphi_n|||_{T_k}^2 + \cdots + |||\varphi_n|||_{T_k}^2 \le \rho[|||D_t^k \varphi_{n-1}|||_{T_k}^2 + \cdots + |||\varphi_{n-1}|||_{T_k}^2]$$

*From this estimate it follows easily that*

$$D_t^i u_n \to D_t^i u \quad \textit{in} \quad \mathcal{E}_t^0(L^2) \quad \textit{as} \quad n \to \infty \quad \textit{for} \quad 0 \le i \le k.$$

Observing equation (6) we can easily conclude that

$$D_t^i D^{3j} u_n \to D_t^i D^{3j} u \quad \text{in} \quad \mathcal{E}_t^0(L^2) \quad \text{as} \quad n \to \infty \quad \text{for} \quad i+j \le k$$

Therefore we obtain following local existence theorem.

**Theorem 5.** *We assume Assumption 2. If*

$$f(x) \in \mathcal{E}_{L^2}^{3(k+1)}$$
$$g(x,t) \in \mathcal{E}_t^{k+1}(L^2) \cap [\mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})]$$

then the Cauchy problem for the *KdV* equation has unique solution $u(x,t)$ in $0 \le t \le T_k$ such that

$$u(x,t) \in \mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})$$
$$|||D_t^i D^{3j} u|||_{T_k} \le \text{const.} \quad \text{for} \quad i+j \le k+1.$$

Combine this local existence theorem with the global a priori estimate (Theorem 4) we can easily conclude the global existence theorem (Theorem 1). This completes the proof of the main theorem.

**Remark.** Uniqueness of the solutions is easily obtained by the usual method of $L^2$-energy estimate.

## References

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