121. A Remark on Singular Integral Operators and Reflection Principle for Some Mixed Problems

By Sadao MIYATAKE

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1969)

§1. Introduction. In this note we consider at first the singular integral operators whose symbols are not necessarily smooth along some hypersurfaces in R^n (Theorem 1). We apply its results to some hyperbolic mixed problems. And we show how to derive the finiteness of propagation speed for these problems. As one might observe, our method of the proof of Theorem 1 is due essentially to A.P. Carderon-A. Zygmund, but the obtained results can be applied directly to some mixed problems.

As for mixed problems, S. Mizohata [4] treated some hyperbolic equations of higher order, and the author [3] showed an extention to the case of fourth order, imposing the assumption only on the boundary. At that time this assumptions clarified the type of the equations suitable to the boundary conditions imposed in [4]. Then K. Asano and T. Shirota [1] investigated the singular integral operators attached to the same type boundary conditions in a half space, and treated the equation (E) below. Now let us remark that Holmgren's transformations (5.1) at the boundary of the equation in [3] and [4] yield such a type of equations as (E), and (E) is the closed form with respect to (5.1). By virtue of Theorem 1 we can apply the reflection principle to the equation (E). This principle makes the treatment in [1] fairly simple. Finally we shall show the finiteness of the propagation speed of the solution, using Lemmas 2, 3 (Theorem 2). The detailed proof will be given in a forthcoming paper.

§2. Singular integral operator. Hereafter we follow the notation of [2]. First of all, let us define the following class of functions.

Definition. A function $h(x)$ defined in R^n is said to be piecewise in $\mathscr{B}^{1+\alpha}$ relative to given hypersurfaces S, if $h(x)$ has the properties: (i) $h(x)$ is continuous in R^n. (ii) $h(x)$ is in $C^{1+\alpha}(\bar{\omega})$, where ω is any connected component of R^n-S. $(0<\alpha<1)$

Theorem 1. *Assume that $h(x, \xi)$ defined in $R^n \times (R^n-\{0\})$ be a C^∞ function of homogeneous degree zero and be piecewise in $\mathscr{B}^{1+\alpha}$ relative to hypersurfaces S with respect to x. Then, for the singular integral operators H, H_1 and H_2 with such symbols $h(x, \xi)$, $h_1(x, \xi)$ and $h_2(x, \xi)$ respectively, we have the following facts:* $H\Lambda-\Lambda H$, $H^*\Lambda-\Lambda H^*$,

$(H^*-H^\#)\Lambda$, $\Lambda(H^*-H^\#)$, $(H_1\circ H_2-H_1H_2)\Lambda$ and $\Lambda(H_1\circ H_2-H_1H_2)$ are bounded operators in L^p.

The proof of this theorem is reduced essentially to the following lemma.

Lemma 1. *Assume that $c(x)$ is piecewise in $\mathcal{B}^{1+\alpha}$ relative to the hyperplane $x_n=0$. Let the symbol $\mathcal{F}[Y](\xi)$ of a singular integral operator T be of spherical mean zero and independent of x. Then for $f(x)$ in $\mathcal{E}^1_{L^p-h}(R^n)$ we have*

$$\|(c(x)T-Tc(x))f_{x_i}\|_p \leqq C_p\|f\|_p, \qquad \text{for } i=1,2,\cdots,n.$$

Proof. (2.1) $\displaystyle\int_{|x-y|>\varepsilon}(c(x)-c(y))Y(x-y)f_{y_i}(y)dy$

has a pointwise limit almost everywhere as ε tends to zero (see [2]). Let us show that L^p norm of (2.1) can be estimated independently of ε by $c\|f\|_p$. By integration by parts (2.1) is equal to

$$\int_{|x-y|=\varepsilon}(c(x)-c(y))Y(x-y)f(y)\gamma_i dS_y-\int_{|x-y|>\varepsilon}c_{y_i}(y)Y(x-y)f(y)dy$$
$$+\int_{|x-y|>1}(c(x)-c(y))Y_{y_i}(x-y)f(y)dy+\int_{1>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f(y)dy.$$

The first and the third terms can be estimated by the method of Hausdorff-Young, and the estimate of the second one is well-known. Now let us decompose $f(y)$ as follows:

$$f(y)=f_1(y)+f_2(y), \qquad f_1(y)=\begin{cases} f(y), & y_n>0 \\ 0, & y_n<0,\end{cases} \quad f_2(y)=\begin{cases} 0, & y_n>0 \\ f(y), & y_n<0.\end{cases}$$

Then the fourth term is equal to

$$I_1(x)+I_2(x)=\int_{1>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f_1(y)dy$$
$$+\int_{1>|x-y|>\varepsilon}(c(x)-c(y))Y_{y_i}(x-y)f_2(y)dy.$$

Now assume that $x=(x_1,\cdots,x_{n-1},x_n)$, $x_n>0$. In the integrand of the first term, $c(x)-c(y)$ is written in the following form:

$$c(x)-c(y)=\sum_{i=1}^n(x_i-y_i)c_{x_i}(x)+b(x,y), \qquad |b(x,y)|\le c|x-y|^{1+\alpha}, \qquad y_n>0.$$

Remark that $\|f_1\|_p\le\|f\|_p$ and that the surface integral of $z_jY_{z_i}(z)$ equals zero. Then $\tilde{I}_1(x)$ defined by $I_1(x)$ in $x_n>0$ and zero in $x_n<0$, can be estimated as in [2]. Concerning $I_2(x)$ we decompose

$$c(x)-c(y)=c(x)-c(x^0)+c(x^0)-c(y)$$
$$=(c(x)-c(x^0))+\sum_{j=1}^{n-1}(x_j-y_j)c_{x_j}(x^0)-y_nc_{x_n}(x^0)+b(x^0,y),$$

where $y_n<0$, $x^0=(x_1,\cdots,x_{n-1},0)$, $|b(x^0,y)|\le c|x^0-y|^{1+\alpha}$.

Then it suffices to discuss the following terms

$$J_1(x)=\int_{1>|x-y|>\varepsilon}-y_nY_{y_i}(x-y)f_2(y)dy$$
$$J_2(x)=\int_{1>|x-y|>\varepsilon}(c(x)-c(x^0))Y_{y_i}(x-y)f_2(y)dy.$$

Take the absolute value of the integrand of $J_1(x)$ and we have

$$|J_1(x)|\leq c\int_{1>|x-y|>\varepsilon}\frac{-y_n}{|x-y|^{n+1}}|f_2(y)|dy\leq c\int_{1>|x-y|>\varepsilon}\frac{x_n-y_n}{|x-y|^{n+1}}|f_2(y)|dy.$$

Similarly from $|c(x)-c(x^0)|\leq cx_n$, the inequality

$$|J_2(x)|\leq c\int_{1>|x-y|>\varepsilon}\frac{x_n}{|x-y|^{n+1}}|f_2(y)|dy\leq c\int_{1>|x-y|>\varepsilon}\frac{x_n-y_n}{|x-y|^{n+1}}|f_2(y)|dy$$

follows. Taking account that the surface integral of $\frac{z_n}{|z|^{n+1}}$ on $|z|=1$ is zero, we can obtain the desired estimates. For $x_n<0$ we get the estimate in the same way.

Remark. If $c(x)$ is not smooth along some hypersurfaces S instead of the hyperplane $x_n=0$, then we may take the corresponding x^0's on S as the points which give the minimal distances from x to the piecewise smooth components of S.

Following the process of [2], we have Theorem 1. q.e.d.

§3. Mixed problem. Consider the following regularly hyperbolic equation in a half space $R^n_+=\{x\,;\,x=(x',x_n)=(x_1,\cdots,x_{n-1},x_n),x_n>0\}$. Denote $x'=(x_1,\cdots,x_{n-1})$.

$$\text{(E)}\quad \left(\frac{\partial^{2m}}{\partial t^{2m}}+\sum_{i=1}^{2m}b_i(x,D)\frac{\partial^{2m-i}}{\partial t^{2m-i}}\right)u+B\left(x,\frac{\partial}{\partial t},D\right)u=f(x,t),$$

where $b_{2m}(x,D)$ is a uniformly elliptic operator. And in each $b_j(x,D)=\sum_{k+|\alpha|=j}a_{k\alpha}(x)\left(\frac{\partial^k}{\partial x_n^k}\right)\left(\frac{\partial}{\partial x'}\right)^\alpha$, $a_{k,\alpha}(x',x_n)$ vanish on $x_n=0$, if k is odd. Let us take the following simple boundary conditions:

$$\text{(BI)}\quad \left(\frac{\partial}{\partial x_n}\right)^{2k}u\bigg|_{x_n=0}=0,\quad k=0,1,\cdots,m-1,$$

$$\text{(BII)}\quad \left(\frac{\partial}{\partial x_n}\right)^{2k+1}u\bigg|_{x_n=0}=0,\quad k=0,1,\cdots,m-1.$$

Theorem 2. *Assume that the coefficients of the principal parts of* (E) *be piecewise in* $\mathcal{B}^{1+\alpha}$ *relative to some hypersurfaces S in* R^n. *Then there exist a unique solution of* (E) *satisfying* (BI) *or* (BII) *for any* $f(x,t)$ *in* $\mathcal{E}^1_t(L^2)$[1] *and any initial data satisfying* (BI) *or* (BII) *respectively. The energy inequality holds in* L^2*-sense. Moreover the solution has a finite speed of propagation just like in the case of Cauchy problem.*

Remark. *In a general domain* Ω *with smooth boundary* Γ *in* R^n, *we can construct the solution of mixed problem for the equation with the same restrictions on* Γ *as* (E), *satisfying the following boundary condition*

1) c.f. [3] p. 288.

(BI)′ $\left\{n(x, D)\right\}^{2k} u\Big|_{\Gamma}=0$ *or* (BII)′ $\left\{n(x, D)\right\}^{2k+1} u\Big|_{\Gamma}=0, \ k=0,1,2, \cdots, m-1.$
Here $n(x, D)=\sum_{i=1}^{n} m_i(x)\frac{\partial}{\partial x_i}$, $(m_1, \cdots, m_n)$ *defined in* Ω *being transversal on* S. *In that proof, the finiteness of the propagation speed described in Theorem* 2 *plays the important role. Then we must take a very nice partition of unity*[2] *on* $\bar{\Omega}$. *The solution in* Ω *has also the finite speed.*

§4. **Reflection principle.** Let us extend the coefficients $a_{k,\alpha}(x', x_n)$ of (E) by the folowing rule: $a_{k,\alpha}(x', x_n)=a_{k,\alpha}(x', x_n)$ if k is even. $a_{k,\alpha}(x', -x_n)=-a_{k,\alpha}(x', x_n)$ if k is odd. Then we can reduce the principal parts of (E) to the following evolution equation (E′)

(E′) $\frac{d}{dt} U=iH\Lambda U+BU+F=AU+F$, where H is an operator of type described in Theorem 1. Consider (E′) in the following two Hilbert spaces

$$\mathcal{H}_1=\left\{U \in \prod^{2m} L^2(R_n),\ U(x', -x_n)=-U(x'\ x_n)\right\},$$

$\mathcal{H}_2=\left\{U \in \prod^{2m} L^2(R^n),\ U(x', -x_n)=U(x', x_n)\right\}$. We introduce in each $\mathcal{H}_i$ a new norm: $\|U\|_{\mathcal{H}_i}=\|\mathcal{N}U\|_{L^2(R^n)}+\beta\|(\Lambda+1)^{-1}U\|_{L^2(R^n)}$, where $\mathcal{N}$ is a digonalizer of $\sigma(H)$ (c.f. [5]). We take the definition domain of A as $\mathcal{D}(A)_i=\mathcal{H}_i \cap \prod^{2m} \mathcal{E}^1_{L^2}(R^n)$, $i=1, 2$. Using Hille-Yosida's theorem and the energy inequality we can arrive at the existence and the uniqueness theorem.

§5. **Finiteness of the propagation speed.** Let us define Holmgren's transformation at the boundary point $x^0=(x_1^0, \cdots, x_{n-1}^0, 0)$ by:

$$t'=t+\sum_{j=1}^{n-1}(x_j-x_j^0)+x_n^2, \quad y_j=x_j, \quad j=1, \cdots, n. \tag{5.1}$$

Then the boundary is also transformed to $y_n=0$. We have

Lemma 2. *By* (5.1), $\left(\frac{\partial}{\partial x_n}\right)^{2k}$ *and* $\left(\frac{\partial}{\partial x_n}\right)^{2k+1}$ *are transformed to the differential operators of even and odd respectively at the boundary.*

Now denote the interior of the backward cone as follows:

$$C_\lambda(x_0, t_0)=\{(x, t) \in R_+^n \times (0\ \infty): (t_0-t) \geq \lambda|x-x_0|\}$$

Lemma 3. *Every point of* $C_\lambda\ (a, \lambda) \cap \{t>0\} \cap \{x_n=0\}$ *contained in the interior of* $C_\lambda(a^0, r\lambda) \cap \{t>0\} \cap \{x_n=0\}$, *where* $a=(a_1, a_2, \cdots, a_{n-1}, a_n)$ $a^0=(a_1, \cdots, a_{n-1}, 0)$, $0<a_n<1$, $r=(1-a_n^2)^{1/2}$.

2) At first we take a suitable partition of unity on $\Gamma: \Sigma\eta_j(s)=1$. We extend $\eta_j(x)$ to some boundary patches ω_j, with the help of the theorem on the systems of ordinary differential equation and the theory of the implicit function. After that we can make $\eta_j(x)$ satisfy $n(x, D)\ \eta_j(x)=0$ in ω_j.

By Lemma 2 and the energy inequality we can see that the solution is locally unique. Now assume that the initial data is zero in $C_{\lambda_0}(a, t_0) \cap \{t=0\}$, where $\lambda_0=\lambda_{\max}^{-1}$: $\lambda_{\max}=\sup_{|\xi|=1\ 1\leq j\leq 2m}\{\lambda_j(x, \xi)\}$, λ_j are the characteristic roots of (E). Using F. John's sweeping out method attached to the parabolic surfaces with tops on (a_0, t), $0<t<rt_0$, and by Lemma 2 we can show that the solution is identically zero in $C_{\lambda_0}(a_0, rt_0) \cap \{t>0\}$. Then considering Lemma 3, we can prove that the solution vanishes identically in $C_{\lambda_0}(a, t_0) \cap \{t>0\}$. From the above arguement we can see that the solution has a finite speed.

References

[1] K. Asano and T. Shirota: On certain mixed problem for hyperbolic equations of higher order. Proc. Japan Acad., **45** (3), 145–148 (1969).

[2] A. P. Carderon and A. Zygmund: Singular integral operators and differential equations. Amer. J. Math., **79**, 901–921 (1957).

[3] S. Miyatake: On some mixed problems for fourth order hyperbolic equations. J. Math. Kyoto Univ., **8** (2), 285–311 (1968).

[4] S. Mizohata: Quelques problèms au bord, du type mixte, pour des équations hyperbolique. Collège de France, 23–60 (1966–1967).

[5] ——: Systèmes hyperboliques. J. Math. Soc. Japan, **11**, 205–233 (1959).