

101. Paracompactness of Product Spaces

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§ 1. As for the normality of the product of a topological space X with a paracompact space Y , several sufficient conditions have been obtained. Indeed, as far as I know, the following are all the cases for which the product $X \times Y$ has been proved to be paracompact for any paracompact space Y .

(a) X is a paracompact space which is a countable union of locally compact closed subsets (K. Morita [6]).¹⁾

(b) X is the image under a closed continuous mapping of a locally compact, paracompact, Hausdorff space (T. Ishii [1]).

(c) X is a regular space which has two coverings $\{C_\gamma | \gamma \in \Gamma\}$ and $\{U_\gamma | \gamma \in \Gamma\}$ such that

(i) C_γ is compact, U_γ is open and $C_\gamma \subset U_\gamma$ for each $\gamma \in \Gamma$, and

(ii) $\{U_\gamma | \gamma \in \Gamma\}$ is order locally finite²⁾ (Y. Katuta [2]).

All spaces considered in this paper are assumed to be Hausdorff. Ishii has showed that his result (b) is not covered by Morita's result (a). On the other hand, Katuta has showed that his result (c) covers Morita's result (a) (in the case when X and Y are regular spaces) and that his result is contained neither in Morita's result nor in Ishii's. Moreover, Katuta stated that he did not know whether his result covers Ishii's result (cf. [2], p. 616).

In this note we shall answer this question negatively in § 2. In § 3 we shall give a sufficient condition for X to possess the property that the product space $X \times Y$ be paracompact for any paracompact space Y , such that all the conditions mentioned above are obtained as special cases of our condition.

§ 2. Let M be a compact Hausdorff space such that $M - \{m_0\}$ is not paracompact for some distinguished point m_0 . Let $\{M_\alpha | \alpha \in A\}$ be an uncountable collection of copies of M . We denote by P the topological sum of $\{M_\alpha\}$. Let X be the quotient space of P which is obtained by identifying all copies of m_0 , and let f be the natural mapping

1) Morita has assumed that X and Y are paracompact normal spaces instead of paracompact Hausdorff spaces.

2) A collection $\{A_\lambda | \lambda \in \wedge\}$ of subsets of a topological space is called order locally finite, if we can introduce a total order $<$ in the index set \wedge such that for each $\lambda \in \wedge$ $\{A_\mu | \mu < \lambda\}$ is locally finite at each point of A_λ .

of P onto X .³⁾ Then we have the following.

Theorem 1. *The space X constructed above satisfies the condition (b), but does not the condition (c).*

Proof. We shall prove only the second part of the theorem, as the first part follows readily from the construction of X . We denote by x_0 the point of X which is obtained by identifying all copies of m_0 . We assume that X satisfies the condition (c), that is, there exist two coverings $\mathfrak{C} = \{C_\gamma | \gamma \in I\}$ and $\mathfrak{U} = \{U_\gamma | \gamma \in I\}$ such that (i) C_γ is compact, U_γ is open and $C_\gamma \subset U_\gamma$ for each $\gamma \in I$, and (ii) $\{U_\gamma | \gamma \in I\}$ is order locally finite.

We put

$$I' = \{\gamma | C_\gamma \ni x_0, \gamma \in I\},$$

then we shall show that I' is uncountable. For this purpose we assume that I' is countable. Then we have $\{\alpha | P' \cap M_\alpha = \{m_0\} \neq \emptyset\}$ where $P' = \bigcup_{\gamma \in I'} f^{-1}(C_\gamma)$. For some point α_0 of $\{\alpha | P' \cap M_\alpha = \{m_0\}\}$ we put

$$\begin{aligned} I_{\alpha_0} &= \{\gamma | f^{-1}(C_\gamma) \cap [M_{\alpha_0} - \{m_0\}] \neq \emptyset, \gamma \in I'\}, \\ \mathfrak{U}_{\alpha_0} &= \{[M_{\alpha_0} - \{m_0\}] \cap f^{-1}(U_\gamma) | \gamma \in I_{\alpha_0}\} \quad \text{and} \\ \mathfrak{C}_{\alpha_0} &= \{[M_{\alpha_0} - \{m_0\}] \cap f^{-1}(C_\gamma) | \gamma \in I_{\alpha_0}\}. \end{aligned}$$

Then $M_{\alpha_0} - \{m_0\}$ has two coverings \mathfrak{U}_{α_0} and \mathfrak{C}_{α_0} which satisfy the condition (c). Hence $M \times [M_{\alpha_0} - \{m_0\}]$ is paracompact. But as is well known, $M \times [M_{\alpha_0} - \{m_0\}]$ is not normal (cf. [6] or [7]). Thus I' is uncountable. Therefore $\{U_\gamma | U_\gamma \ni x_0, \gamma \in I\}$ is uncountable. This contradicts the order local finiteness of $\{U_\gamma | \gamma \in I\}$. Thus the proof is completed.

The existence of a space M with the property described above is seen from an example below.

Example. Let Ω be the first uncountable ordinal. We take $\{\alpha | \alpha \leq \Omega\}$ with order topology and Ω for M and m_0 in the above construction respectively. Then the space which is constructed from $\{\alpha | \alpha \leq \Omega\}$ as established above, satisfies the condition described in Theorem 1.

§ 3. Theorem 2. *Let Y be a paracompact space and let $\{K_\lambda | \lambda \in A\}$ be a discrete family of compact subsets of a collectionwise normal space X . If for any closed subset F of X which is contained in $X - \bigcup_{\lambda \in A} K_\lambda$ the product $F \times Y$ is paracompact, then $X \times Y$ is paracompact.*

For the proof we shall first prove the following lemma.

Lemma. *Let Y be a paracompact space and let x_0 be a point of a regular space X . If for any subset F of X contained in $X - \{x_0\}$ $F \times Y$ is paracompact, then $X \times Y$ is paracompact.*

Proof. Let \mathfrak{B} be any open covering of $X \times Y$. Then there exist

3) This process is due to Ishii's example in [1].

4) \emptyset denotes the empty set.

a family $\{U_\gamma | \gamma \in \Gamma\}$ of neighborhoods of x_0 in X and a locally finite open covering $\{V_\gamma | \gamma \in \Gamma\}$ of Y such that

- (i) $\{U_\gamma \times V_\gamma | \gamma \in \Gamma\}$ covers $\{x_0\} \times Y$, and
- (ii) for any $\gamma \in \Gamma$ there exists $W \in \mathfrak{B}$ such that $U_\gamma \times V_\gamma \subset W$.

Moreover, we can select a family $\{U'_\lambda | \lambda \in A\}$ of neighborhoods of x_0 and a locally finite open covering $\{V'_\lambda | \lambda \in A\}$ of Y such that

- (i') $\{U'_\lambda \times V'_\lambda | \lambda \in A\}$ covers $\{x_0\} \times Y$, and
- (ii') $\overline{\{U'_\lambda \times V'_\lambda | \lambda \in A\}} \subset \bigcup \{U_\gamma \times V_\gamma | \gamma \in \Gamma\}$.

Then $[X - U'_\lambda] \times \bar{V}'_\lambda$ is paracompact by the assumption of the theorem and $X_1 = \bigcup \{[X - U'_\lambda] \times \bar{V}'_\lambda | \lambda \in A\}$ is paracompact by [3]. Hence there exists a locally finite open covering \mathfrak{B}_1 of X_1 which refines $\{W \cap X_1 | W \in \mathfrak{B}\}$. Then $\left\{W_1 \cap \left[\bigcup_{\lambda \in A} \{[X - \bar{U}'_\lambda] \times V'_\lambda\} \right] \mid W_1 \in \mathfrak{B}_1 \right\}$ is a locally finite family of open subsets of X . It is easily proved that $\left\{W_1 \cap \left[\bigcup_{\lambda \in A} \{[X - \bar{U}'_\lambda] \times V'_\lambda\} \right] \mid W_1 \in \mathfrak{B}_1 \right\} \cup \{U_\gamma \times V_\gamma | \gamma \in \Gamma\}$ is a locally finite open covering of X and refines \mathfrak{B} by (ii'). This completes the proof.

Now we shall return to the proof of Theorem 2.

Proof of Theorem 2. Let X_1 be a quotient space of X which is obtained by contracting K_λ to a point x_λ for every $\lambda \in A$. We denote by f the natural mapping of X onto X_1 , then f is a perfect mapping. $\{x_\lambda | \lambda \in A\}$ is a closed discrete subset of the collectionwise normal space X_1 . Hence there exists a discrete collection $\{U_\lambda | \lambda \in A\}$ of open subsets of X_1 such that $U_\lambda \ni x_\lambda$ for every $\lambda \in A$. By lemma and perfectness of f $\bar{U}_\lambda \times Y$ is paracompact for every $\lambda \in A$ and also by the assumption of the theorem we can easily prove that $\left[X_1 - \bigcup_{\lambda \in A} U_\lambda \right] \times Y$ is paracompact.

Therefore $X_1 \times Y = \bigcup_{\lambda \in A} \{ \bar{U}_\lambda \times Y | \lambda \in A \} \cup \left[X_1 - \left(\bigcup_{\lambda \in A} U_\lambda \right) \right] \times Y$ is paracompact by [3]. Let $g: X \times Y \rightarrow X_1 \times Y$ be a mapping such that $g(x, y) = (f(x), y)$ for every $(x, y) \in X \times Y$; then g is a perfect mapping. Hence $X \times Y$, which is an inverse image under g of paracompact space $X_1 \times Y$, is paracompact. Thus the proof is completed.

In case X is paracompact we have the following.

Theorem 3. *Let Y be a paracompact space and let K be a locally compact closed subset of a paracompact space X . If for any closed subset F of X contained in $X - K$, $F \times Y$ is paracompact, then $X \times Y$ is paracompact.*

Proof. By the assumption of the theorem there exists a locally finite family $\mathfrak{K} = \{K_\lambda | \lambda \in A\}$ of compact subsets of X such that $\bigcup \{K_\lambda | \lambda \in A\} = K$. For every $x \in X$ we can select a neighborhood $U(x)$ of x such that $U(x)$ meets only finite number of elements of \mathfrak{K} . Let $\mathfrak{B} = \{V_\gamma | \gamma \in \Gamma\}$ be a locally finite open covering of X such that for every

$\gamma \in \Gamma$ \bar{V}_γ is contained in some $U(x)$. By Theorem 1 $\bar{V}_\gamma \times Y$ is paracompact for every $\gamma \in \Gamma$. Thus $X \times Y = \bigcup \{\bar{V}_\gamma \times Y \mid \gamma \in \Gamma\}$ is paracompact by [3]. The proof is completed.

Theorem 4. *If there exist a locally compact closed subset K of a paracompact space X such that any closed subset F of X contained in $X - K$ has the condition (c), then $X \times Y$ is paracompact for any paracompact space Y .*

This is obvious from Theorem 3. In case (b) X becomes paracompact and is the union of a discrete closed set K and a locally compact subset $X - K$ (cf. [4]). Hence any closed subset of $X - K$ has the condition (c). Therefore case (b) is included in Theorem 4. Moreover, it is easy to see that the condition in Theorem 4 holds for case (c).

References

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