100. On q-Spaces

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(Comm. by Kinjirô KUNUGI, M. J. A., June 10, 1969)

1. Introduction. The notion of q-spaces has been introduced by E. Michael [4]. A topological space Y is called a q-space if every $y \in Y$ has a sequence $\{N_i\}$ of neighborhoods satisfying the following condition (q):

$$\text{(q)} \quad \begin{cases} \text{If } \{y_i\} \text{ is an infinite sequence of points in } Y \text{ such that} \\ y_i \in N_i \text{ for each } i, \text{ then } \{y_i\} \text{ has an accumulation point} \\ \text{in } Y. \end{cases}$$

An M-space is clearly a q-space, but the converse does not hold. Indeed, a locally compact space is a q-space but not always an M-space (cf. K. Morita [5]). These two notions of spaces, however, are connected by the notion of almost open mappings due to P. Vopěnka [6]. Namely we shall prove

Theorem 1.*) *A regular space Y is a q-space if and only if there exists an almost open continuous mapping f from a regular M-space onto Y.*

The definition of almost open mapping is as follows: A mapping $f: X \to Y$ is called almost open if for any point y of Y there is a point $x \in f^{-1}(y)$ having a basis of open sets such that the image of each member of the basis is open in Y. This notion of mappings is also available for a characterization of pointwise countable type in the sense of A. Arhangel'skii [1] by paracompact M-spaces.

Theorem 2. *A topological space Y is of pointwise countable type if and only if there exists an almost open continuous mapping f from a paracompact M-space X onto Y.*

Recently it has been shown by T. Isiwata [3] that the product of M-spaces need not be an M-space. By a counter example given by him, we see also that the product of q-spaces is not necessarily a q-space. In the final section we shall give a sufficient condition for the product of q-spaces to be a q-space.

2. Proof of Theorem 1. Let Y be a regular q-space and let y be a point of Y. Then y has a sequence $\{N_i\}$ satisfying condition (q) and $\bar{N}_{i+1} \subset N_i$ for each i. Let us put $K = \cap \{\bar{N}_i : i = 1, 2, \cdots\}$. Then each sequence in K has an accumulation point in K, since $\{N_i\}$ satisfies

*) Cf. J. Nagata: Mappings and M-spaces, Proc. Japan Acad., **45**, 140–144 (1969), which appeared after the preparation of this paper.

condition (q). Therefore K is countably compact. Moreover $\{N_i\}$ is a countable base for K in the sense of Arhangel'skii (cf. [1, p. 36]). To prove this, let O be any open set in Y and suppose that $O \not\supset N_i$ for each i. In case $N_i - 0$ consists of a finite number of points for some i, there is a point $p \in \cap\{N_i - O; i=1, 2, \cdots\}$ since $N_i - O \neq \emptyset$ for each i. Hence $K \not\subset O$. In case $N_i - O$ consists of an infinite number of points for each i, we can choose an infinite sequence $\{y_i\}$ such that $y_i \in N_i - O$ for each i. By condition (q), $\{y_i\}$ has an accumulation point p in Y. Obviously $p \in K$ and $p \notin O$. Thus $K \not\subset O$. Hence $\{N_i\}$ is a countable base for K. Consequently there exists a covering $\{K_\alpha; \alpha \in \Omega\}$ of Y by countably compact sets of countable character.

Now, for each $\alpha \in \Omega$, let us consider the new space Y_α that consists of all the points in Y. The topology for Y_α is defined as follows: Each point of $Y_\alpha - K_\alpha$ is isolated and neighborhoods of a point y in K_α are the same as in Y. The regularity of Y_α can be easily verified. Let $\{U^i_\alpha; i=1, 2, \cdots\}$ be a countable base for K_α in Y such that $U^{i+1}_\alpha \subset U^i_\alpha$ for each i and put $\mathfrak{U}^i_\alpha = \{U^i_\alpha\} \cup \{\{y\}; y \in Y_\alpha - U^i_\alpha\}$ for $i=1, 2, \cdots$. Then $\{\mathfrak{U}^i_\alpha; i=1, 2, \cdots\}$ is a normal sequence in Y_α satisfying the condition (M) in [5]. Accordingly Y_α is a regular M-space.

Finally, let X be the topological sum of $\{Y_\alpha; \alpha \in \Omega\}$ and let $f: X \to Y$ be the natural surjection; $f \mid Y_\alpha: Y_\alpha \to Y$ is the identity as a map of sets. Then X is clearly a regular M-space. To show that f is almost open, let y be any point of Y. If we choose $\alpha \in \Omega$ such that $y \in K_\alpha$, then $f^{-1}(y) \cap Y_\alpha$ consists of a single point which we denote by x. Let $\mathfrak{U}$ be a neighborhood basis at x in Y. Then $\mathfrak{V} = \{f^{-1}(U) \cap Y_\alpha; U \in \mathfrak{U}\}$ is a neighborhood basis at x in X and $f(V)$ is open in Y for each $V \in \mathfrak{V}$. The continuity of f is trivial. Thus the "only if" part is proved.

The proof of the "if" part follows from the next lemma.

Lemma 1. *An almost open image of a q-space is a q-space.*

Proof. Let X be a q-space and let $f: X \to Y$ be an almost open continuous mapping. Let y be a point of Y. Then there is a point $x \in f^{-1}(y)$ having a basis $\mathfrak{U}$ of open sets such that $f(U)$ is open in Y for each $U \in \mathfrak{U}$. Since X is a q-space, $\mathfrak{U}$ contains a sequence $\{U_i\}$ satisfying condition (q). Then, as is easily shown, $\{f(U_i)\}$ is a sequence of neighborhoods of y satisfying condition (q). Hence Y is a q-space.

Remark. Theorem 1 remains true if we replace the word "regular" by "completely regular" or "normal" throughout.

3. Proof of Theorem 2. The "only if" part can be proved similarly as that of Theorem 1.

The proof of "if" part follows from the next lemma, since a paracompact M-space is of pointwise countable type (cf. [1, Theorem 3.10]).

Lemma 2. *An almost open image of a regular space of pointwise countable type is of pointwise countable type.*

Proof. Let X be a regular space of pointwise countable type and let $f ; X \to Y$ be an almost open continuous mapping. Let y be a point of Y. Then there is a point $x \in f^{-1}(y)$ having a basis $\mathfrak{U}$ of open sets such that $f(U)$ is open in Y for each $U \in \mathfrak{U}$. Since X is of pointwise countable type, there is a countable base $\{V_i\}$ for a compact set containing x. Let us choose a sequence $\{U_i\}$ from the members of $\mathfrak{U}$ such that $\bar{U}_{i+1} \subset U_i \cap V_i$ for each i. Then $\{U_i\}$ is a countable base for the compact set $K = \cap \{\bar{U}_i\}$ and thus $\{f(U_i)\}$ is a countable base for the compact set $f(K)$ containing y. Hence Y is of pointwise countable type.

4. Products of q-spaces. Let us consider the following condition (k_0):

(k_0) $\begin{cases}\text{Each sequence containing an accumulation point has a} \\ \text{subsequence which is contained in a compact set.}\end{cases}$

Evidently k-spaces and sequential compact spaces satisfy condition (k_0). From the proof of Theorem 1 we have immediately the following theorem.

Theorem 3. *A regular space Y is a q-space satisfying condition* (k_0) *if and only if there exists an almost open continuous mapping f from a regular M-space X satisfying condition* (k_0) *onto Y.*

In Ishii, Tsuda, and Kunugi [2] the class $\mathfrak{C}$ of spaces is defined. By our condition (k_0) the class $\mathfrak{C}$ is expressed as follows: An M-space belongs to the class $\mathfrak{C}$ if and only if it satisfies condition (k_0). Thus their results read as follows:

1) The product of an M-space satisfying condition (k_0) and an arbitrary M-space is an M-space.

2) The product of a countable number of M-spaces satisfying condition (k_0) is again an M-space satisfying condition (k_0).

Since the product of a countable number of almost open mappings is almost open, the next theorem follows at once from the above facts and Theorem 3.

Theorem 4. 1) *The product of a regular q-space satisfying condition* (k_0) *and an arbitrary regular q-space is a regular q-space.*

2) *The product of a countable number of regular q-spaces satisfying condition* (k_0) *is again a regular q-space satisfying condition* (k_0).

References

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