50. *Some Properties of Regular Distribution Semi-groups**)

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The notion of distribution semi-groups was defined by J. L. Lions in [4]. D. Fujiwara characterized the infinitesimal generator of an exponential distribution semi-group in terms of an equi-continuous semi-group in some Fréchet space in [3]. In this paper we shall report that Fujiwara's results can be partially extended to the infinitesimal generator of a regular distribution semi-group if we introduce the notion of a locally equi-continuous semi-group, which was recently studied by T. Kōmura in [3]. In order to characterize the infinitesimal generator of a locally equi-continuous semi-group in a locally convex space, she used the concept of the generalized Laplace transform of a distribution. This concept is also essential to the present work. Complete proofs of the theorems in the present note will be published elsewhere.

§1. Statements of the results. Let $L(E, F)$ be the totality of continuous linear mappings from E to F, where E and F are topological vector spaces. The set $L(E, E)$ is denoted by $L(E)$. Let us abbreviate Schwartz space $\mathcal{D}(R^1)$ by $\mathcal{D}$. Let X be a Banach space. Consider the totality of X-valued distributions, $\mathcal{D}'(X)=L(\mathcal{D}, X)$. Let $\mathcal{D}_+$(or $\mathcal{D}'_+(X)$) be the totality of elements of $\mathcal{D}$(or $\mathcal{D}'(X)$) whose supports are contained in $[0, \infty)$. For any linear operator T, we denote its domain (or range, or null space) by $D(T)$ (or $R(T)$, or $N(T)$).

Following Lions, we say that an $L(X)$-valued distribution $\mathcal{T}$ is a regular distribution semi-group (D.S.G., in short) if $\mathcal{T}$ satisfies the following five conditions.

(T.1) $\mathcal{T} \in \mathcal{D}'_+(L(X))$.

(T.2) $\mathcal{T}(\phi*\psi)=\mathcal{T}(\phi)\mathcal{T}(\psi)$ *if* $\phi, \psi \in \mathcal{D}_+$.

(T.3) $\bigcap_{\phi\in\mathcal{D}_+} N(\mathcal{T}(\phi))=\{0\}$.

(T.4) *The linear hull* $\mathcal{R}$ *of* $\bigcup_{\phi\in\mathcal{D}_+} R(\mathcal{T}(\phi))$ *is dense in* X.

(T.5) *For any* $x \in \mathcal{R}$, *there exists an* X*-valued function* $x(t)$ *such that*: (i) $x(t)=0$ *for* $t<0$, (ii) $x(0)=x$, (iii) $x(t)$ *is continuous for* $t\geqq 0$, (iv) $\mathcal{T}(\phi)x=\int_0^\infty \phi(t)x(t)dt$ *for any* $\phi \in \mathcal{D}$.

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Let $\mathcal{T}$ be a D.S.G. For any distribution F with compact support, we can define an operator $\mathcal{T}(F)$ in X by the relation:

$$\mathcal{T}(F)x \equiv \sum_{j=1}^{n} \mathcal{T}(F*\phi_j)y_j \qquad \text{for } x=\sum_{j=1}^{n} \mathcal{T}(\phi_j)y_j \in \mathcal{R}.$$

By Peetre's argument in [5], $\mathcal{T}(F)$ is uniquely determined, densely defined and pre-closed. The closure of $\mathcal{T}(-\delta')$[1] is called the infinitesimal generator of $\mathcal{T}$. For it we can prove the following theorems.

Theorem 1. *Let A be the infinitesimal generator of a D.S.G. $\mathcal{T}$. Then A has the following properties.*

(A.1) *For any integer $n \geqq 1$, A^n is a densely defined closed operator.*

(A.2) *The set $Y \equiv \bigcap_{n=0}^{\infty} D(A^n)$ is dense in X.*

(A.3) *A is the closure of the restriction $A|_Y$ of A to Y.*

(A.4) *The set Y becomes a Fréchet space with respect to the semi-norm system $\{\|x\|_n = \|A^n x\|\,;\, n=0, 1, \cdots\}$. The operator $A|_Y$ generates the locally equi-continuous semi-group T_t in Y which satisfies the condition:*

(1.1) $\mathcal{T}(\phi)x = \int_0^\infty \phi(t) T_t x\, dt$ *for any $x \in Y$ and any $\phi \in \mathcal{D}$.*

Theorem 2. *Assume that a linear operator A satisfies the conditions* (A.1) *to* (A.4). *If A has nonempty resolvent set, then A is the infinitesimal generator of a D.S.G. $\mathcal{T}$ which satisfies the condition* (1.1).

§2. The well-posedness of the Cauchy problem in the sense of distribution. First we recall the facts concerning the generalized Laplace transform of distribution, which was discussed by T. Kōmura in [3]. Let $\hat{\phi}(\lambda) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} \phi(t) dt$ for complex λ and $\phi \in \mathcal{D}$. Then $\hat{\phi}(\lambda)$ is an entire function. Let $\boldsymbol{D} \equiv \{\hat{\phi} : \phi \in \mathcal{D}\}$. $\boldsymbol{D}$ can be considered as a locally convex topological vector space such that the mapping: $\phi \to \hat{\phi}$ is a homeomorphism from $\mathcal{D}$ to $\boldsymbol{D}$. Let $\boldsymbol{D}'(X) \equiv L(\boldsymbol{D}, X)$. There is a homeomorphism: $f \to \hat{f}$ from $\mathcal{D}'(X)$ to $\boldsymbol{D}'(X)$, defined by the relation: $\hat{f}(\hat{\phi}) \equiv f(\phi)$ for any $\phi \in \mathcal{D}$. The functional $\hat{f}$ is called the generalized Laplace transform of f. Let $\boldsymbol{D}'_+(X) \equiv \{\hat{f} : f \in \mathcal{D}'_+(X)\}$. It is a closed subspace of $\boldsymbol{D}'(X)$ since $\mathcal{D}'_+(X)$ is the closed subspace of $\mathcal{D}'(X)$, and it is homeomorphic to $\mathcal{D}'_+(X)$ by the mapping: $\hat{f} \to f$. Let F be a distribution, and let $x \in X$. Define $f \equiv F \otimes x$ by the equality: $(F\otimes x)(\phi) = F(\phi)x$ for any $\phi \in \mathcal{D}$. It is clear that $f \in \mathcal{D}'(X)$ (or $\mathcal{D}'_+(X)$) if F is a distribution (or a distribution with supp (F)[2] $\subset [0, \infty)$), and that $\hat{f} = \hat{F} \otimes x$. For example, we have $(\delta \otimes x)^\wedge = \hat{\delta} \otimes x = 1 \otimes x$.

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1) We denote by δ' the differentiation of the Dirac's delta δ.

2) Here supp (F) is the support of the distribution F.

For any linear operator A such that $D(A)\cup R(A)\subset X$, we define an operator $\boldsymbol{A}$ in $\boldsymbol{D}'_+(X)$ induced by A through the relations:

$$D(\boldsymbol{A})\equiv\{\hat{f} : f\in\mathcal{D}'_+(X),\ f(\phi)\in D(A) \text{ for any } \phi\in\mathcal{D}, \text{ the mapping } g \text{ defined by } g(\phi)\equiv A(f(\phi)) \text{ belonging to } \mathcal{D}'_+(X).\},$$

$$\boldsymbol{A}\hat{f}\equiv\hat{g} \text{ for any } \hat{f}\in D(\boldsymbol{A}).$$

We also define an operator $\boldsymbol{\lambda}\in L(\boldsymbol{D}'_+(X))$ by the relation:

$$(\boldsymbol{\lambda}\hat{f})(\phi)\equiv\hat{f}(\lambda\hat{\phi})=f(-\phi')=\left(\frac{d}{dt}f\right)(\phi).^{3)}$$

Then $\boldsymbol{\lambda}$ is a homeomorphism from $\boldsymbol{D}'_+(X)$ to $\boldsymbol{D}'_+(X)$.

If $(\boldsymbol{\lambda}-\boldsymbol{A})$ has the inverse $(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}$ belonging to $L(\boldsymbol{D}'_+(X))$, then the operator A is said to be well-posed for the Cauchy problem at $t=0$ in the sense of distribution (well-posed, in short). As for the well-posedness of A, we have the following lemmas.

Lemma 1. *An operator A is well-posed if and only if A is closed and there exists $\mathcal{T}\in\mathcal{D}'_+(L(X))$ satisfying conditions:*

(2.1) *For any $x\in X$ and any $\phi\in\mathcal{D}$, $\mathcal{T}(\phi)x\in D(A)$ and*

$$\left[\frac{d}{dt}\mathcal{T}(\phi)\right]x-A[\mathcal{T}(\phi)x]=\delta(\phi)x.^{3)}$$

(2.2) *For any $x\in D(A)$ and any $\phi\in\mathcal{D}$, $\mathcal{T}(\phi)Ax=A\mathcal{T}(\phi)x$.*

In this case $\mathcal{T}$ satisfies the relation:

(2.3) *For any $x\in X$ and any $\phi\in\mathcal{D}$, $\mathcal{T}(\phi)x=(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}(1\otimes x)(\hat{\phi})$.*

Lemma 2. *If A is the infinitesimal generator of a D.S.G., then A is well-posed.*

Lemma 3. *Let A be well-posed. Then we have:* (i) $\mathcal{T}\in\mathcal{D}'_+(L(X))$ *($\mathcal{T}(\phi)x$ is defined by* (2.3)), (ii) *for any integer $n\geqq 1$, A^n is a closed operator satisfying $D(A^n)\supset\mathcal{R}$ ($\mathcal{R}$ is defined in* (T.4)), (iii) *the statement* (A.4) *is true.*

Lemma 1 is the fundamental tool. The necessity of conditions follows from the relations: $(\boldsymbol{\lambda}-\boldsymbol{A})(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}\hat{f}=\hat{f}$ for any $\hat{f}\in\boldsymbol{D}'_+(X)$ and $(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}(\boldsymbol{\lambda}-\boldsymbol{A})\hat{f}=\hat{f}$ for any $\hat{f}\in D(\boldsymbol{A})$. The sufficiency of them follows from the fact that the algebraic tensor product $\mathcal{D}'_+\otimes X$ is dense in $\mathcal{D}'_+(X)$. Lemma 2 is due to Lions [4, Theorem 4.1]. One can prove the assertion verifying the relations (2.1) and (2.2). The crucial point in proof of Lemma 3 is the following fact. If $\mathcal{T}\in\mathcal{D}'_+(L(X))$, then for any $a>0$, there exist an integer $n=n_a$ and an $L(X)$-valued continuous function $T_a(t)$ with $\text{supp}\,(T_a)\subset[0,\infty)$ such that: $\mathcal{T}(\phi)=\int_{-\infty}^{\infty}(-1)^n\phi^{(n)}(t)T_a(t)dt$ for any $\phi\in\mathcal{D}$ with $\text{supp}\,(\phi)\subset(-\infty,a]$.

§3. Outline of proof of theorems. Let A be the infinitesimal generator of a D.S.G. $\mathcal{T}$. Then, by definition, we have that $\mathcal{R}$ is

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3) The symbol '$\frac{d}{dt}$' means the differentiation in the sense of distribution.

dense in X, and that A is the closure of $A|_Y$. Therefore, combining Lemmas 2 and 3, we get Theorem 1.

Let A be well-posed. Then $\mathcal{T}$ (defined by (2.3)) satisfies (T.1) and (T.3) by definition, and satisfies (T.2) and (T.5) by Lemma 3. Assume further that $D(A)$ is dense in X. We can show that the adjoint operator of A is well-posed using Lemma 1, and that $\mathcal{T}$ satisfies (T.4). Therefore $\mathcal{T}$ is a D.S.G. Let B be the infinitesimal generator of $\mathcal{T}$. We can show $\boldsymbol{A}=\boldsymbol{B}$ from Lemmas 1 and 2. This implies $A=B$. Namely we have:

Theorem 3. *A linear operator A in X is the infinitesimal generator of a D.S.G. if and only if A is densely defined and well-posed. In this case the generated D.S.G. $\mathcal{T}$ is represented by*

$$\mathcal{T}(\phi)x=(\boldsymbol{\lambda}-\boldsymbol{A})^{-1}(1\otimes x)(\hat{\phi}) \text{ for any } x\in X \text{ and any } \phi\in\mathcal{D}.$$

Since A has a nonempty resolvent set, we can conclude from (A.4) that for any fixed $a>0$ there exists an integer n such that $\|\mathcal{T}(\phi)x\| \leqq \text{const} \sup_{|t|\leqq a, 0\leqq j\leqq n} |\phi^{(j)}(t)|\,\|x\|$ for any $\phi\in\mathcal{D}$ with $\text{supp}\,(\phi)\subset[-a, a]$ and any $x\in Y$. Therefore $\mathcal{T}$ can be extended as an operator belonging to $\mathcal{D}'(L(X))$. From (1.1) we have for $x\in Y$ and $\phi\in\mathcal{D}$, $\frac{d}{dt}\mathcal{T}(\phi)x-A\mathcal{T}(\phi)x =\delta(\phi)x$. From (A.2) and (A.3) we can show the relations (2.1) and (2.2) of Lemma 1. This implies the well-posedness of A. Moreover $D(A)$ is dense by (A.1). Theorem 3 assures Theorem 2.

Remark. This April Professor J. L. Lions kindly informed the author the work of Mr. J. Chazarain which appeared in C. R. Acad. Sci. Paris Sér. A, t., **266**, 10–13 (1968). He had characterized a well-posed operator by its resolvent. Especially he proved that a well-posed operator has a nonempty resolvent set. Taking account of his result, we can say that the conditions of Theorem 2 characterize the infinitesimal generator of a D. S. G.

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