2. A Note on the Metrizability of M-Spaces

By Harold R. BENNETT
Texas Technological College

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 13, 1969)

The notion of an M-space was introduced by Morita in [6] and in [1] Okuyama gave conditions for an M-space to be metrizable. Recently Borges, in [2], generalized some of Okuyama's work by considering $w\Delta$-spaces. In this note a condition is given under which a $w\Delta$-space is a Moore space.

The terminology of [4] will be used except all spaces will be T_1.

Definition 1. A space X is said to be a $w\Delta$-space if there exists a sequence $\{B_1, B_2, \cdots\}$ of open covers of X such that for each $x_0 \in X$, if $x_n \in \mathrm{St}(x_0, B_n)$ for each natural number n, then the sequence $\{x_1, x_2, \cdots\}$ has a cluster point.

Definition 2. A space X is said to be an M-space provided there exists a normal sequence[1] of open coverings of X satisfying the following: If $\{A_1, A_2, \cdots\}$ is a sequence of subsets of X with the finite intersection property and if there exists $x_0 \in X$ such that for each natural number n there exists some $A_k \subset \mathrm{St}(x_0, B_n)$, then

$$\bigcap_{i=1}^{\infty} A_i^- \neq \emptyset.$$

Clearly all metrizable or countably compact spaces are M-spaces. In [2], Borges shows that each M-space is also an $w\Delta$-space.

Definition 3. Let X be a regular space. Then X is a quasi-developable space if there exists a sequence $\{G_1, G_2, \cdots\}$ of collections of open subsets of X such that if $x \in X$ and R is an open subset of X containing x, then there is a natural number $n(x, R)$ such that some element of $G_{n(x,R)}$ contains x and each member of $G_{n(x,R)}$ that contains x lies in R. The sequence $\{G_1, G_2, \cdots\}$ is called the quasi-development for X.

Notice that if, in Definition 3, it is also required that each G_i be a cover for X, then X satisfies the first three parts of Axiom 1 of [5] and X is called a Moore space. In this case $\{G_1, G_2, \cdots\}$ is called a development for X.

Quasi-developable spaces are investigated extensively in [1] where

1) A sequence $\{U_1, U_2, \cdots\}$ of open covers of a topological space X is a normal sequence if for each natural number n $\mathrm{St}(x, U_{n+1})$ is contained in some element of U_n, for each $x \in X$.

it is shown that a quasi-developable space that has closed sets G_δ[2] is a Moore space (see Theorem 1 of [1]).

Definition 4. A topological space is said to be pointwise paracompact if each open covering has a point finite open refinement.

Theorem 1. *A pointwise paracompact $w\Delta$-space is a Moore space if and only if it is a quasi-developable space.*

Proof. Let X be a pointwise paracompact, quasi-developable $w\Delta$-space with quasi-development $\{G_1, G_2, \cdots\}$ and let $\{B_1, B_2, \cdots\}$ be a sequence of open covers for X such that if $x_0 \in X$ and $x_n \in \mathrm{St}(x, B_n)$ for each natural number n, then the sequence $\{x_1, x_2, \cdots\}$ has a cluster point.

Let M be any closed set in X. Construct a sequence of open sets whose common part is M in the following fashion. If $x \in M$, let $x(1)$ be the first natural number such that $\mathrm{St}(x, G_{x(1)}) \neq \emptyset$. Choose $g(x, 1) \in G_{x(1)}$ such that $x \in g(x, 1)$ and let

$$a(x, 1) = g(x, 1) \cap \mathrm{St}(x, B_1).$$

Then

$$A_1 = \{a(x, 1) : x \in M\}$$

is an open covering of M and, since X is pointwise paracompact, there exists a point finite (in X) open refinement H_1 of A_1. Suppose $H_1, H_2, \cdots, H_{k-1}$ have been defined. If $x \in M$, let $x(k)$, if it exists, be the first natural number greater than $x(k-1)$ such that

(i) $\mathrm{St}(x, G_{x(k)}) \neq \emptyset$, and

(ii) $\mathrm{St}(x, G_{x(k)})^{-} \subset \cap \{h \in H_{k-1} : x \in h\}$.

Notice that $\cap \{h \in H_{k-1} : x \in h\}$ is an open set since H_{k-1} is a point finite collection of open sets. If $i \leq k$ and $\mathrm{St}(x, G_i) \neq \emptyset$, then choose one member of G_i that contains x. Designate this member of G_i by $g(x, i)$. Let

$$c(x, k) = \cap \{g(x, i) : i \leq k,\ \mathrm{St}(x, G_i) \neq \emptyset\}.$$

If no such natural number $x(k)$ exists, then let

$$c(x, k) = c(x, k-1).$$

Let

$$a(x, k) = c(x, k) \cap \mathrm{St}(x, B_k).$$

Then

$$A_k = \{a(x, k) : x \in M\}$$

is an open covering of M and there exist an open point finite (in X) refinement H_k. Define H_j for each natural number j. It follows that

$$M = \bigcap_{i=1}^{\infty} H_i^*$$

where

$$H_i^* = \cup \{h \in H_i\}.$$

2) A topological space X has closed sets G_δ if each closed subset of X can be expressed as the countable intersection of open sets.

For suppose there exists $z \in X$ such that

$$z \in \bigcap_{i=1}^{\infty} H_i^* - M.$$

Then let

$$H_i(z) = \{h \in H_i : z \in h\}.$$

Notice that for each natural number i, $H_i(z)$ is a finite set. For each $h \in H_i(z)$ choose one element $x(h)$ of M such that h refines $a(x(h), i)$. Let

$$L_i(z) = \{a(x(h), i) : h \in H_i(z)\}.$$

Then, for each natural number i, $L_i(z)$ if a finite set and if $\lambda \in L_{i+1}(z)$, then there exists $\lambda' \in L_i(z)$ such shat $\lambda' \supset \lambda^-$. Thus $\{L_1(z), L_2(z), \cdots\}$ satisfies the conditions of Theorem 114 of [5]. Hence, for each natural number i, there exists $a(x_i, i) \in L_i(z)$ such that

$$a(x_i, i) \supset a(x_{i+1}, i+1)^-.$$

Since $z \in a(x_i, i) = c(x_i, i) \cap \mathrm{St}(x_i, B_i)$, it follows that $x_i \in \mathrm{St}(z, B_i)$ for each natural number i. Thus $\{x_1, x_2, \cdots\}$ has a cluster point p and since $x_i \in M$, for each i, $p \in M$. Thus $p \neq z$. It is also clear that

$$p \in \bigcap_{i=1}^{\infty} a(x_i, i).$$

Since X is a quasi-developable space there is a natural number N_1 such that

(i) $\mathrm{St}(p, G_{N_1}) \neq \emptyset$, and

(ii) $z \notin \mathrm{St}(p, G_{N_1})^-$.

Because p is a cluster point of $\{x_1, x_2, \cdots\}$ there is a natural number N_2 such that for all $j \geq N_2$, $x_j \in \mathrm{St}(p, G_{N_1})$. Let $N = \max\{N_1, N_2\}$. Then

$$X_N \in \mathrm{St}(p, G_{N_1})$$

and

$$p \in c(x_N, N).$$

By construction

$$c(x_N, N) = \cap \{g(x_N, i) : i \leq N\}$$

and since $N_1 \leq N$ it follows that

$$C(x_N, N) \subseteq g(x_N, N_1) \subseteq \mathrm{St}(p, G_{N_1}).$$

But this cannot be so because $z \in c(x_N, N)$. Thus $M = \bigcap_{i=1}^{\infty} H_i^*$ and, by Theorem 1 of [1], X is a Moore space.

The converse is obvious.

Corollary 1. *A pointwise paracompact, quasi-developable $w\Delta$-space has a uniform base.*[3]

Proof. R. W. Heath in [7] has shown that a pointwise paracompact Moore space has a uniform base.

Corollary 2. *A pointwise paracompact quasi-developable M-space is a Moore space.*

---

3) A base B of a topological space X is a uniform base if, for $x \in X$, any infinite subset of B, each member of which contains x, is a base at x.

Corollary 3. *A paracompact, quasi-developable $w\Delta$-space is metrizable.*

Proof. A paracompact Moore space is metrizable.

References

[1] H. R. Bennett: Quasi-developable Spaces. Arizona State University Dissertation (1968).
[2] Carlos J. R. Borges: On metrizability of topological spaces. Can. Math. J., **20**, 795–804 (1968).
[3] R. W. Heath: Screenability, pointwise paracompactness, and metrization of Moore spaces. Can Math. J., **16**, 763–770 (1964).
[4] J. K. Kelley: General Topology (Van Nostrand, Princeton, N. J., 1955).
[5] R. L. Moore: Foundations of point set theory. Revised edition, Amer. Math. Soc. Coll. Pub. XIII (1962).
[6] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).
[7] A. Okuyama: On metrizability of M-spaces. Proc. Japan Acad., **40**, 176–179 (1964).