169. *On Lacunary Trigonometric Series. II*

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§1. Introduction. In [3] we have proved

Theorem A. *Let $\{n_k\}$ be a sequence of positive integers and $\{a_k\}$ a sequence of non-negative real numbers for which the conditions*

$$(1.1)\qquad n_{k+1}>n_k(1+ck^{-\alpha}),\qquad k=1, 2, \cdots,$$

$$(1.2)\qquad A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}\to+\infty,\qquad as\ N\to+\infty,$$

and

$$(1.3)\qquad a_N=o(A_N N^{-\alpha}),\qquad as\ N\to+\infty,$$

are satisfied, where c and α are any given constants such that

$$(1.4)\qquad c>0\qquad and\qquad 0\leq\alpha\leq 1/2.$$

Then we have, for all x,

$$(1.5)\qquad \lim_{N\to\infty}|\{t\,;\,t\in E,\ \sum_{k=1}^{N} a_k\cos 2\pi n_k(t+\alpha_k)\leq xA_N\}|\,/\,|E| = (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du,^{*)}$$

where $E\subset[0, 1]$ is any given set of positive measure and $\{\alpha_k\}$ any given sequence of real numbers.

This theorem was first proved by R. Salem and A. Zygmund in case of $\alpha=0$, where $\{n_k\}$ satisfies the so-called *Hadamard's gap* condition (cf. [4], (5.5), pp. 264–268). In that case they also remarked that under the hypothesis (1.2) the condition (1.3) is *necessary* for the validity of (1.5) (cf. [4], (5.27), pp. 268–269).

Further, in [2] P. Erdös has pointed out that for every positive constant c there exists a sequence of positive integers $\{n_k\}$ such that $n_{k+1}>n_k(1+ck^{-1/2})$, $k\geq 1$, and (1.5) is not true for $a_k=1$, $k\geq 1$, and $E=[0, 1]$. But I could not follow his argument on the example.

The purpose of the present note is to prove the following

Theorem B. *For any given constants $c>0$ and $0\leq\alpha\leq 1/2$, there exist sequences of positive integers $\{n_k\}$ and non-negative real numbers $\{a_k\}$ for which the conditions* (1.1), (1.2) *and*

$$(1.6)\qquad a_N=O(A_N N^{-\alpha}),\qquad as\ N\to+\infty,$$

are satisfied, but (1.5) *is not true for $E=[0, 1]$ and $\alpha_k=0$, $k\geq 1$.*

The above theorem shows that in Theorem A the condition (1.3) is

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*) $|E|$ denotes the Lebesgue measure of a set E.

indispensable for the validity of (1.5). In §§ 3–5 we prove Theorem B for $0<\alpha\leq 1/2$.

§ 2. Some lemmas. i. In this section let $\{X_k(\omega)\}$ be a sequence of independent random variables on some probability space $(\Omega, \mathfrak{B}, P)$ with vanishing mean values and finite variances. Putting $E(X_k^2)=\sigma_k^2$ and $s_m^2=\sum_{k=1}^{m}\sigma_k^2$, the theorem of Lindeberg reads as follows:

Theorem. (Cf. [1] pp. 56–57.) *Under the hypotheses*

$$s_m\to+\infty \quad \text{and} \quad \sigma_m=o(s_m), \quad \text{as } m\to+\infty, \tag{2.1}$$

a necessary and sufficient condition for the validity of the relation

$$\lim_{m\to\infty} P\{s_m^{-1}\sum_{k=1}^{m} X_k(\omega)\leq x\}=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du \tag{2.2}$$

for all x is that, for any given $\varepsilon>0$,

$$\lim_{m\to\infty} s_m^{-2}\sum_{k=1}^{m}\int_{X_k>\varepsilon s_m} X_k^2(\omega)dP(\omega)=0.$$

From the above theorem the following lemma is easily seen.

Lemma 1. *Under the hypotheses* (2.1), *the relation* (2.2) *implies that, for any given $\varepsilon>0$,*

$$\lim_{m\to\infty}\sum_{k=1}^{m} P\{X_k(\omega)>\varepsilon s_m\}=0. \tag{2.3}$$

ii. **Lemma 2.** *Let m and l be any given positive integers. Then there exists a positive constant c_0, not depending on m and l, such that*

$$|\{t\,;\,t\in[0,1],\ |\sum_{j=0}^{l}\cos 2\pi m(j+1)t|>(l+1)/3\}|\geq 2c_0 l^{-1}.$$

Proof. This can be easily seen from the relation

$$\sum_{j=0}^{l}\cos 2\pi m(j+1)t=\frac{\sin 2\pi m(l+3/2)t}{2\sin \pi mt}-1/2,$$

provided if $\sin \pi mt\neq 0$.

§ 3. Construction of sequences. In the following let $c>0$ and $0<\alpha\leq 1/2$ be given constants in Theorem B. First let us put

$$\begin{cases} p(j)=[j^{1/\alpha}], \\ l(j)=\operatorname{Min}\{[p^\alpha(j)/c],\ p(j+1)-p(j)-1\}, \\ j_0=\operatorname{Min}\{j\,;\,l(j)\geq 1\}.^{*)} \end{cases} \tag{3.1}$$

Since $p(j+1)-p(j)\sim\alpha^{-1}j^{(1-\alpha)/\alpha}$ and $p^\alpha(j)\sim j$, as $j\to+\infty$,**) we have

$$l(j)\sim\beta(\alpha)j, \qquad \text{as } j\to+\infty, \tag{3.2}$$

where

$$\beta(\alpha)=\begin{cases} 1/c, & \text{if } 0<\alpha<1/2, \\ \operatorname{Min}(2,\ 1/c), & \text{if } \alpha=1/2. \end{cases} \tag{3.3}$$

Next we put

$$n_1=1 \quad \text{and} \quad n_{k+1}=[n_k(1+ck^{-\alpha})+1], \quad \text{for } k+1<p(j_0).$$

If $n_{p(j)}$, $j\geq j_0$, is defined, then we put

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*) $[x]$ denotes the integral part of x.

**) $f(j)\sim g(j)$, as $j\to+\infty$, means that $f(j)/g(j)\to 1$, as $j\to+\infty$.

$$n_{p(j)+l}=\begin{cases} n_{p(j)}(1+l), & \text{if } 1\leq l\leq l(j), \\ [n_{p(j)+l-1}\{1+cp^{-\alpha}(j)\}+1], & \text{if } l(j)<l<p(j+1)-p(j). \end{cases}$$

Further we put, for $j\geq j_0$,

(3.4) $$n_{p(j)}=2^{q(j)},$$

where

(3.5) $$q(j)=\begin{cases} \mathrm{Min}[m\,;2^m/n_{p(j_0)-1}>1+c\{p(j_0)-1\}^{-\alpha}, & \text{if } j=j_0, \\ \mathrm{Min}[m\,;2^m/n_{p(j)-1}>1+c\{p(j)-1\}^{-\alpha} & \\ \qquad \text{and } 2^m>n_{p(j-1)}j^3], & \text{if } j>j_0. \end{cases}$$

Then it is clear that the sequence $\{n_k\}$ satisfies (1.1).

On the other hand we define $\{a_k\}$ as follows:

(3.6) $$a_k=\begin{cases} 1, & \text{if } p(j)\leq k\leq p(j)+l(j), \quad \text{for some } j\geq j_0, \\ 1/k^2, & \text{if otherwise.} \end{cases}$$

Then we have, by (3.6) and (3.2),

(3.7) $$A^2_{p(m)+l(m)}=2^{-1}\sum_{j=j_0}^{m}\{l(j)+1\}+O(1)\sim\beta(\alpha)m^2/4, \quad \text{as } m\to+\infty.$$

Since $p^\alpha(j)\sim j$, as $j\to+\infty$, this sequence $\{a_k\}$ satisfies both of the conditions (1.2) and (1.6).

Further, if we put $S_N(t)=\sum_{k=1}^{N}a_k\cos 2\pi n_k t$, then we have, by the definitions of $\{n_k\}$ and $\{a_k\}$, uniformly in t,

(3.8) $$S_{p(m)+l(m)}(t)=\sum_{j=j_0}^{m}\sum_{l=0}^{l(j)}\cos\{2\pi 2^{q(j)}(1+l)t\}+O(1), \quad \text{as } m\to+\infty.$$

§4. **Independent functions.** Let $x_k(t)$ be the k-th digit of the infinite *dyadic* expansion of t, $0\leq t\leq 1$, that is,

(4.1) $$t=\sum_{k=1}^{\infty}x_k(t)2^{-k}, \qquad (x_k(t)=0 \text{ or } 1),$$

then $\{x_k(t)\}$ is a sequence of independent functions on the interval $[0, 1]$. Putting

(4.2) $$\eta_j(t)=\sum_{k=q(j)}^{q(j+1)-1}x_k(t)2^{-k}, \qquad \text{for } j\geq j_0,$$

we define

(4.3) $$\begin{cases} \mu_j=\int_0^1\sum_{l=0}^{l(j)}\cos 2\pi 2^{q(j)}(l+1)\eta_j(t)dt, \\ Y_j(t)=\sum_{l=0}^{l(j)}\cos\{2\pi 2^{q(j)}(l+1)\eta_j(t)\}-\mu_j, \\ \tau_j^2=\int_0^1 Y_j^2(t)dt \quad \text{and} \quad t_m^2=\sum_{j=j_0}^{m}\tau_j^2. \end{cases}$$

Then we have, by (4.2) and (3.5),

$$\begin{aligned} &\sup_t\Big|\sum_{l=0}^{l(j)}\cos\{2\pi 2^{q(j)}(l+1)t\}-\sum_{l=0}^{l(j)}\cos 2\pi\{2^{q(j)}(l+1)\eta_j(t)\}\Big| \\ &=O(\sup_t 2^{q(j)}\sum_{k=q(j+1)}^{\infty}x_k(t)2^{-k}\sum_{l=0}^{l(j)}(l+1)) \\ &=O(2^{q(j)}j^2\sum_{k=q(j+1)}^{\infty}2^{-k}) \\ &=O(2^{q(j)-q(j+1)}j^2)=O(j^{-1}), \qquad \text{as } j\to+\infty. \end{aligned}$$

Therefore, we have

(4.4) $$\sup_t |Y_j(t)-\sum_{l=0}^{l(j)}\cos 2\pi 2^{q(j)}(l+1)t|=O(j^{-1}), \qquad \text{as } j\to+\infty,$$

and, by (3.2) and (3.7),

(4.5) $$t_m^2=2^{-1}\sum_{j=j_0}^{m}(l(j)+1)^2+O(\log m)$$
$$\sim A^2_{p(m)+l(m)}\sim\beta(\alpha)m^2/4, \qquad \text{as } m\to+\infty.$$

Thus we obtain

(4.6) $$t_m\to+\infty \quad\text{and}\quad \tau_m=o(t_m), \qquad \text{as } m\to+\infty.$$

Further, we have, by (3.8) and (4.4),

(4.7) $$|S_{p(m)+l(m)(t)}-\sum_{j=j_0}^{m}Y_j(t)|=O(\log m)$$
$$=o(A_{p(m)+l(m)}), \text{ uniformly in } t, \text{ as } m\to+\infty.$$

Since (4.5) and (3.2) imply that $\sqrt{\beta(\alpha)}\,t_m/5<\{l([m/2])+1\}/4$, for $m\ge m_0$, we have, by (4.4)

$$\sum_{j=m/2}^{m}|\{t\,;0\le t\le 1,\ |Y_j(t)|>\sqrt{\beta(\alpha)}\,t_m/5\}|$$
$$\ge\sum_{j=m/2}^{m}|[t\,;0\le t\le 1,\ \sum_{l=0}^{l(j)}\cos 2\pi 2^{q(j)}(l+1)t>\{l(j)+1\}/3]|,$$

and by Lemma 2, we have

(4.8) $$\varliminf_{m\to\infty}\sum_{j=m/2}^{m}|\{t\,;0\le t\le 1,\ |Y_j(t)|>\sqrt{\beta(\alpha)}\,t_m/5\}|$$
$$\ge\varliminf_{m\to\infty}2c_0\sum_{j=m/2}^{m}l^{-1}(j)\ge\varliminf_{m\to\infty}c_0\beta(\alpha)^{-1}\sum_{j=m/2}^{m}j^{-1}>0.$$

§5. **Proof of Theorem B.** Suppose that (1.5) holds, that is,

(5.1) $$\lim_{m\to\infty}|\{t\,;0\le t\le 1,\ A^{-1}_{p(m)+l(m)}S_{p(m)+l(m)}(t)\le x\}|$$
$$=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du.$$

Then by (4.7) and (4.5), we have

(5.2) $$\lim_{m\to\infty}|\{t\,;0\le t\le 1,\ t_m^{-1}\sum_{j=j_0}^{m}Y_j(t)\le x\}|$$
$$=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du.$$

On the other hand (4.1), (4.2), and (4.3) imply that $\{Y_j(t)\}$ is a sequence of independent functions with vanishing mean values and finite variances. By (5.2) and (4.6) we can apply Lemma 1 to $\{Y_j(t)\}$ and obtain

$$\lim_{m\to\infty}\sum_{j=m/2}^{m}|\{t\,;0\le t\le 1,|Y_j(t)|>\sqrt{\beta(\alpha)}\,t_m/5\}|=0.$$

This contradicts with (4.8).

References

[1] H. Cramér: Random Variable and its Probability Distribution. Cambridge Tract (1962).

[2] P. Erdös: On trigonometric sums with gaps. Magyar Tud. Akad. Kutatós Int. Közl., **7**, 37–42 (1962).
[3] S. Takahashi: On lacunary trigonometric series. Proc. Japan Acad., **41**, 503–506 (1965).
[4] A. Zygmund: Trigonometric Series. Vol. II. Cambridge University Press (1959).