145. *Almost Convergent Topology*

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1. **Introduction.** The study in function space topologies mainly has been investigated in the space of continuous functions (see [3]). Recently Kolmogorov [2], Prokhorov [4], and Skorokhod [5] discussed topologies on the space of all discontinuous functions of the first kind in connection with a problem in probability theory. In the theory of probability, if the independent variable t is considered to be the time, then it is impossible to assume the existence of an instrument which will measure time exactly whence a comparatively weaker topology is considered (see [5]). In this paper for the above mentioned purpose almost convergent topology is considered and in the end of this paper one shows Skorokhod M-convergent is a special case of almost convergent topology.

(2.1) **Definition.** Let (X, L) and (Y, S) be topological spaces. For each pair of open sets $U \in L$ and $V \in S$, let

$$A(U, V) \equiv \{f \in Y^X : f(U) \cap V \neq \phi\}.$$

An almost convergent topology on Y^X is that topology which has as subbasis $\{A(U, V)\}$.

The following example provides a motivation to study the topology.

(2.2) **Example.** Let $f_n : [0, 1] \to R$ (reals with the usual topology)

$$f = \begin{cases} 0 & \text{on } [0, 1/2), \\ 1 & \text{on } [1/2, 1]. \end{cases}$$

$$f_n = \begin{cases} 0 & \text{on } [0, 2^n - 1/2^{n+1}), \\ 2^n x - (2^n - 1)/2 & \text{on } [(2^{n-1})/(2^{n+1}), (2^n + 1)/(2^{n+1})), \\ 1 & \text{on } [2^n + 1/2^{n+1}, 1] \end{cases}$$

for $n = 1, 2, \cdots$.

Since $f_n \notin P(1/2, S_\nu(1)) = \{f \in Y^X : f(1/2) \subset S_\nu(1)\}$, where $S_\nu(1)$ is the open sphere about 1 with the radius $\nu < 1/2$. $\{f_n\} \not\to f$ in the point open topology. However, $\{f_n\} \to f$ in the almost convergent topology (we denote as $\{f_n\} \overset{A}{\to} f$ from now on). As the relation with other topologies we have

(3.1) **Theorem.** *A-topology (Almost Convergent Topology) $\subset$ P-topology (point open topology).*

Proof. Let $A(U, V)$ be a subbasic open nbhd in A-topology and $f \in A(U, V)$ where $U \in L$, $V \in S$ and L, S, are topologies in the domain and range spaces respectively. Then there exists $x \in U$ such that $f(x) \cap V = \phi$ which implies $f(x) \in V$ and $f \in P(x, V)$. Therefore, $P(x, V)$ is an open nbhd of f and contained in $A(U, V)$.

Combining Example (2.2) and Theorem (3.1) we have [3.2].

Corollary. *Almost convergent topology is strictly smaller than the point open topology except that they coincide when (X, L) is the discrete topology.*

Proof. The first statement is an easy consequence of previous results and if (X, L) is a discrete space then

$$P(x, V) = \{f \in Y^X : f(x) \in V\} = \{f \in Y^X : f(x) \cap V \neq \phi\}$$

and $x \in L$ which implies $P(x, V) = A(x, V)$.

As a separation axiom we have

[3.3] **Theorem.** *The set of all continuous functions on X to Y which is denoted as $C(X)$ is T_1 with respect to the A-topology whenever (Y, S) is Hausdorff.*

Proof. Let $f, g \in C(X)$ and $f \neq g$. Then there exists $x \in X$ such that $p = f(x) \neq g(x) = q$. Since Y is a Hausdorff space there exist $U, V \in S$ with $f(x) \in U$, $g(x) \in V$, and $U \cap V = \phi$. Since $f, g \in C(X)$ there exist $O_1, O_2 \in L$ such that $x \in O_1 \cap O_2$ and

$$f(O_1) \subset U \text{ and } g(O_2) \subset V.$$

Then $(g(O_2) \cap U) \subset (V \cap U) = \phi$ and $(f(O_1) \cap V) \subset (V \cap U) = \phi$. Let $O = O_1 \cap O_2$ then $x \in O \in L$ and $g \notin A(O, U)$, $f \notin A(O, V)$ while $f \in A(O, U)$ and $g \in A(O, V)$.

It is well known that in the point open topology $\lim f_n = f$ iff $\lim f_n(x) = f(x)$ for every $x \in X$. By Corollary [3.2] we expect a wider result in the A-topology. In fact we have the following example.

[4.1] **Example.**

$$f_1 = \begin{cases} 2x & \text{on } [0, 1/2), \\ 2 - 2x & \text{on } [1/2, 1]. \end{cases}$$

$$f_2 = \begin{cases} 2^2 x & \text{on } [0, 1/2^2), \\ 2 - 2^2 x & \text{on } [1/2^2, 1/2), \\ -2 + 2^2 x & \text{on } [1/2, 3/2^2), \\ 2^2 - 2^2 x & \text{on } [3/2^2, 1]. \end{cases}$$

$$f_n = \begin{cases} 2^n x & \text{on } [0, 1/2^n), \\ \vdots & \qquad \vdots \\ 2^n - 2^n x & \text{on } [2^n - 1/2^n, 1]. \end{cases}$$

Since dyadic fractions are dense in $[0, 1]$ $\{f_n\} \xrightarrow{A} f$ where

$$f = \begin{cases} 1 & \text{on rationals}, \\ 0 & \text{on irrationals}. \end{cases}$$

Moreover, let $\tilde{f}=1$ and $f=0$, then $\{f_n\}\xrightarrow{A}\tilde{f}$ and $\{f_n\}\xrightarrow{A}\tilde{\tilde{f}}$ where $\tilde{f}(X)\cap\tilde{\tilde{f}}(X)=\phi$ and both $\tilde{f}$ and $\tilde{\tilde{f}}$ are continuous functions. In fact if $\{f_n\}\xrightarrow{A}f$ and the graph of f is dense in the graph of $\tilde{f}$ then $\{f_n\}\xrightarrow{A}\tilde{f}$ also.

There is an interesting relation between A-topology and Skorokhod M-topology which he denoted as M_2-topology (see [5] p. 266) in the space of all functions which are defined on the interval [0, 1] whose range space Y is a complete separable metric space, and which at every point have a limit on the left and are continuous on the right.

[4.2] **Definition (Skorokhod).**

$$R[(x_1, f(x_1)), (x_2, f(x_2))]=|x_1-x_2|+d(f(x_1), f(x_2))$$

where $d(f(x_1), f(x_2))$ is the distance of $f(x_1)$ and $f(x_2)$ in Y $\{f_n\}$ is said to be M-convergent to f iff

$$\lim_{n\to\infty}\ \sup_{(x_1,f(x_1))\in G(f)}\ \operatorname{Inf}_{(x_2,f_n(x_2))\in G(f_n)} R[(x_1, f(x_1)), (x_2, f_n(x_2))]=O,$$

where $G(f)=\{(x, f(x)) : x\in[0, 1]\}$.

Let $U_n=S(x, 1/n)=\{z:|x-z|<1/n, z\in X\}$ and $V_n=S_d(f(x), 1/n)=\{y: d(f(x), y)<1/n, y\in Y\}$ then $A=(U_n, V_n)$ is an element in the A-topology and $\{f\}\to f$ in the A-topology iff $G(f_l)\cap(U_n\times V_n)=\phi$ at each point $x\in X$ and $l\geq N_x$ for some fixed N_x. i.e., $f_l\in A(U_n, V_n)$ for $l\geq N_x$. Therefore, Skorokhod M-topology is a special case of A-topology.

References

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