233. *A New Characterization of Hausdorff k-Spaces*

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Throughout, we shall assume all topological spaces are Hausdorff. A function $f: X \to Y$ from a space X to a space Y is said to be *weakly-continuous* if and only if $f^{-1}(y)$ is closed in X for each y in Y.

Let $f: X \to Y$ be a function from a space X to a space Y. The following are two properties which a space X may or may not satisfy:

$P_1(X)$: f is weakly continuous;

$P_2(X)$: for any filter base ([2], p. 211) $\{F_\lambda \mid \lambda \in \Lambda\}$ of compact sets of X, we have $f(\bigcap_{\lambda \in \Lambda} F_\lambda) = \bigcap_{\lambda \in \Lambda} f(F_\lambda)$.

Theorem 1. *If X is any space, then $P_1(X)$ implies $P_2(X)$.*

The proof of this theorem is straightforward. To our surprise, we discovered first the following:

Theorem 2. *If X is a k-space, then $P_2(X)$ implies $P_1(X)$; and hence $P_1(X)$ and $P_2(X)$ are equivalent.*

Proof. See the "necessity" part of the proof for Theorem 3, below.

Trying, in vain, to weaken the hypothesis of Theorem 2, we obtain the following characterization of k-spaces.

Theorem 3. *$P_1(X)$ and $P_2(X)$ are equivalent if and only if X is a k-space.*

Proof. *Necessity.* According to a theorem of Cohen [1], (see also [2], p. 248), X is a k-space if and only if it is a quotient space of a locally compact space, say Z. Let $p: Z \to X$ denote the natural projection (=quotient map). Suppose, $P_1(X)$ is false, i.e., there exists an element y in Y such that $f^{-1}(y)$ is not closed in X, then $p^{-1}(f^{-1}(y))$ is not closed in Z. Hence, there exists an element z in $Cl(p^{-1}(f^{-1}(y))$ such that $f(p(z)) \neq y$. Since Z is locally compact (and Hausdorff), there is a filter base $\{E_\lambda \mid \lambda \in \Lambda\}$ of compact neighborhoods E_λ of z such that $\bigcap_{\lambda \in \Lambda} E_\lambda = \{z\}$. Let $F_\lambda = p(E_\lambda)$ for all $\lambda \in \Lambda$, then $\{F_\lambda : \lambda \in \Lambda\}$ is a filter base of compact subsets of X such that $\bigcap_{\lambda \in \Lambda} F_\lambda = \{p(z)\}$. Then we have $f(\bigcap_{\lambda \in \Lambda} F_\lambda) = f(p(z))$; but $\bigcap_{\lambda \in \Lambda} f(F_\lambda)$ contains the element y, which is not in $f(\bigcap_{\lambda \in \Lambda} F_\lambda)$. This shows $f(\bigcap_{\lambda \in \Lambda} F_\lambda) \neq \bigcap_{\lambda \in \Lambda} f(F_\lambda)$, which contradicts $P_2(X)$. Thus, $P_2(X)$ and $P_1(X)$ are equivalent by the preceding and by Theorem 1.

Sufficiency. Assume X is not a k-space. Then there exists F, a non-closed subset of X, such that $F \cap K$ is closed in K for every compact subset K of X. Define $f: X \to X$ as follows:

Let z be any fixed element of F,

$$f(x)=\begin{cases} z \text{ if } x \in F\,; \\ x \text{ if } x \in X-F. \end{cases}$$

Then, $P_1(X)$ is false for the function f, since $f^{-1}(z)=F$. Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a filter base of compact subsets of X. The fact that $f(\bigcap_{\lambda \in \Lambda} K_\lambda) \subseteq \bigcap_{\lambda \in \Lambda} f(K_\lambda)$ is clear. Thus, to show that $P_2(X)$ is true we need only show $\bigcap_{\lambda \in \Lambda} f(K_\lambda) \subseteq f(\bigcap_{\lambda \in \Lambda} K_\lambda)$. Let $p \in \bigcap_{\lambda \in \Lambda} f(K_\lambda)$. If $p=z$ then $K_\lambda \cap F \neq \phi$ for every λ. But since $K_\lambda \cap F$ is closed in K_λ, $\{K_\lambda \cap F \mid \lambda \in \Lambda\}$ is a filter base of compact subsets of X and hence $\bigcap_{\lambda \in \Lambda}(K_\lambda \cap F)=F \cap (\bigcap_{\lambda \in \Lambda} K_\lambda) \neq \phi$. Thus, $p=z$ is contained in $f(\bigcap_{\lambda \in \Lambda} K_\lambda)$. If $p \neq z$, then $f^{-1}(p)=p=f(p)$. Thus, $p=f(p) \in \cap f(K_\lambda)$ implies $p \in K_\lambda$, for every λ. Consequently, $p \in \cap K_\lambda$ and so $f(p) \in f(\cap K_\lambda)$. Hence, $P_1(X)$ and $P_2(X)$ are not equivalent.

Q.E.D.

References

[1] D. E. Cohen: Spaces with weak topology. Quart. J. Math., Oxford Ser., **5**, 77–80 (1953).

[2] J. Dugundji: Topology. Allyn and Bacon, Inc., Boston (1966).