155. *On Some Integral Equations with Normal Integral Operators*

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In the present paper we deal with the construction and the function theoretical properties of solutions of some integral equations with normal integral operators.

Definitions of notations. Let $\varDelta$ be a Lebesgue ρ-measurable set of finite or infinite measure in m-dimensional real Euclidean space R_m; let $L_2(\varDelta, \rho)$ be the Lebesgue functionspace; let $\{\varphi_\nu(x)\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu(x)\}_{\mu=1,2,3,\cdots}$ be both incomplete orthonormal systems such that the union of them forms a complete orthonormal system in $L_2(\varDelta, \rho)$; let $((\beta_{ij}))$ be the bounded normal operator in the Hilbert coordinate space l_2 corresponding to an infinite bounded normal matrix (β_{ij}) with $\sum_{j=1}^{\infty}|\beta_{ij}|^2 \neq |\beta_{ii}|^2>0$ $(i=1, 2, 3, \cdots)$; let $((\beta_{ij}^{(p)}))=((\beta_{ij}))^p$ $(p=1, 2, 3, \cdots, n)$ where $\beta_{ij}^{(1)}=\beta_{ij}$ $(i, j=1, 2, 3, \cdots)$; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be any infinite bounded sequence of complex scalars; and let N_p be integral operators defined by

$$N_p h(x)=\sum_{\nu=1}^{\infty}\lambda_\nu^p\int_\varDelta h(y)\overline{\varphi_\nu(y)}d\rho(y)\cdot\varphi_\nu(x)+c^p\sum_{\mu=1}^{\infty}\left\{\int_\varDelta h(y)\overline{\psi_\mu(y)}d\rho(y)\cdot\sum_{j=1}^{\infty}\beta_{\mu j}^{(p)}\psi_j(x)\right\}$$
$$(p=1, 2, 3, \cdots, n;\ h(x)\in L_2(\varDelta, \rho)),$$

where c is an arbitrarily given complex constant. Then, as we discussed before [1], N_1 is a bounded normal operator in $L_2(\varDelta, \rho)$ and $N_p=N_1^p$.

Theorem 1. *Let $g(x)$ be any given function in $L_2(\varDelta, \rho)$ such that it consists of all of $\varphi_\nu(x)$, $\psi_\mu(x)$; let $\zeta_p(p=1, 2, 3, \cdots, n)$ be the roots of the equation $\lambda^n+\sum_{p=1}^{n}a_p\lambda^{n-p}=0$ with complex coefficients a_p; let $\{\lambda_\nu\}$ be everywhere dense on an open or a closed rectifiable Jordan curve; let $\sup_\nu|\lambda_\nu|>|c|\left\{\sum_{i,j=1}^{\infty}|\beta_{ij}|^2\right\}^{\frac{1}{2}}<\infty$; and let $\sigma=\max_p\{|\zeta_p|\cdot\sup_\nu|\lambda_\nu|\}$. Then the integral equation*

$$(1)\qquad \lambda^n f(x)+\sum_{p=1}^{n}a_p\lambda^{n-p}N_p f(x)=g(x)\qquad (\sigma<|\lambda|<\infty)$$

has a uniquely determined solution

$$(2)\qquad f_\lambda(x)=\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x)+\frac{1}{\lambda^n}\left\{g(x)-\sum_{\nu=1}^{\infty}c_\nu\varphi_\nu(x)\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{1}{\lambda^k}\kappa_k(\zeta_1, \zeta_2, \cdots, \zeta_n)c^k\sum_{\mu=1}^{\infty}(g, \psi_\mu)\sum_{j=1}^{\infty}\beta_{\mu j}^{(k)}\psi_j(x)\right\}\in L_2(\varDelta, \rho),$$

where $c_\nu=(g,\varphi_\nu)=\int_\Delta g(x)\overline{\varphi_\nu(x)}d\rho(x)$ *and* $\kappa_k(\zeta_1,\zeta_2,\cdots,\zeta_n)=\sum_{i,j,\cdots,l=1}^n\zeta_i\zeta_j\cdots\zeta_l$ *for* $i\leqq j\leqq\cdots\leqq l$ *and* $\frac{i}{i}+\frac{j}{j}+\cdots+\frac{l}{l}=k$. *Moreover, if we set* $\chi(\lambda)=(f_\lambda,h)$ *for an arbitrarily given* $h(x)\in L_2(\Delta,\rho)$ *such that it consists of all of* $\varphi_\nu(x)$ *and* $\psi_\mu(x)$, $M(r)=\max_{\theta\in[0,2\pi]}|\chi(re^{i\theta})|$ $(\sigma<r<\infty)$, *and*

$$m(r,\infty)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|\chi(re^{-it})|\,dt\qquad(\sigma<r<\infty),$$

and denote by Γ *a rectifiable closed Jordan curve containing the disc* $|\lambda|\leqq\sigma$ *inside itself, then the function* $\chi(\lambda)$ *enjoys the following properties*:

(A) $$\frac{1}{2\pi i}\int_\Gamma\frac{\chi(\lambda)}{(\lambda-z)^k}d\lambda=\begin{cases}0 & (\textit{for every } z \textit{ inside } \Gamma)\\ -\dfrac{\chi^{(k-1)}(z)}{(k-1)!} & (\textit{for every } z \textit{ outside } \Gamma),\end{cases}$$

where the complex line integral along Γ *is taken counterclockwise*;

(B) $M(r')\leqq M(r)$ *for* $\sigma<r<r'<\infty$ *and* $M(r)\to\infty$ *as* $r\to\sigma$;

(C) $$\frac{1}{2\pi}\int_0^{2\pi}|\chi(re^{it})|^2dt=\sum_{\nu=1}^\infty|a_\nu(r)|^2<\infty$$

$$\sqrt{\sum_{\nu=1}^\infty|a_\nu(r)|^2}\leqq M(r)\leqq\sum_{\nu=1}^\infty|a_\nu(r)|<\infty,$$

where

$$a_\nu(r)=\frac{1}{\pi}\int_0^{2\pi}\chi(re^{it})\cos\nu t\,dt\qquad(\sigma<r<\infty);$$

(D) *if we put* $T(r)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\log\sqrt{1+|\chi(re^{-it})|^2}\,dt$ $(\sigma<r<\infty)$, *then* $T(r)$ *is not only a monotone decreasing function of* r *but also a convex function of* $\log r$;

(E) *if, for any large positive number* G, *there exist a positive constant* ρ_G *in a bounded open interval* (σ,l) $(\sigma<l<\infty)$ *and a set* $A_{\theta(\rho_G)}$, *with positive measure* m_G, *of angles* θ *such that the inequality* $|\chi(\rho_Ge^{-i\theta})|>G$ *holds for every* $\theta\in A_{\theta(\rho_G)}$ *and that* $\inf_G m_G>0$, *then, for uncountably many complex numbers* $\{c_\omega\}$ *chosen suitably,* $\chi(\lambda)$ *has a denumerably infinite number of* c_ω*-points* $b_\mu^{(c_\omega)}$ $(\mu=1,2,3,\cdots)$, *repeated according to the respective orders, in the domain* $D_\sigma\{\lambda:\sigma<|\lambda|<\infty\}$ *such that any accumulation point of them lies on the circle* $|\lambda|=\sigma$ *and that the positive series* $\sum_{\mu=1}^\infty(|b_\mu^{(c_\omega)}|-\sigma)$ *is divergent*;

(F) *if* $\{\zeta_\kappa\}_{\kappa=1,2,3,\cdots,j}$ *are all distinct roots of* $\lambda^n+\sum_{p=1}^n a_p\lambda^{n-p}=0$ *and the order of* ζ_κ *is denoted by* m_κ, *then*

$$\overline{\lim_{r\to\sigma+0}}\frac{m(r,\infty)}{\log[(r-\sigma)^{-1}]}\leqq\sum_{\kappa=1}^j\frac{m_\kappa(m_\kappa+1)}{2}\qquad(\sum_{\kappa=1}^j m_\kappa=n).$$

Proof. It is verified at once that the given integral equation (1) is rewritten in the form $\prod_{p=1}^n(\lambda I-\zeta_pN_1)\cdot f(x)=g(x)$ and moreover that $\|N_1\|=\max\{\sup_\nu|\lambda_\nu|, \ |c|\,\|((\beta_{ij}))\|\}$ [1]. On the other hand, since $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$ by hypothesis, we have for any $\alpha=(\alpha_1, \alpha_2, \alpha_3, \cdots)\in l_2$

$$\begin{aligned}\|((\beta_{ij}))\alpha\|^2&=\sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}\alpha_j\beta_{ij}\right|^2\\ &\leqq\sum_{i=1}^{\infty}\left\{\sum_{j=1}^{\infty}|\alpha_j|^2\cdot\sum_{j=1}^{\infty}|\beta_{ij}|^2\right\}=\sum_{i,j=1}^{\infty}|\beta_{ij}|^2\cdot\|\alpha\|^2<\infty\end{aligned}$$

and hence| $\|((\beta_{ij}))\|\leqq\left\{\sum_{i,j=1}^{\infty}|\beta_{ij}|^2\right\}^{\frac{1}{2}}$. This last inequality and the hypotheses on $\sup_\nu|\lambda_\nu|$ and σ lead us to the result that the solution of(1) is given by

$$\begin{aligned}f_\lambda(x)&=\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}}\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}dK(z)g(x)\qquad(\sigma<|\lambda|<\infty)\\ &=\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x)+\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}-\{\lambda_\nu\}}\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}dK(z)g(x),\end{aligned}$$

where $c_\nu=(g, \varphi_\nu)$ and $\{K(z)\}$ denotes the complex spectral family of N_1. In addition,

$$\begin{aligned}&\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}-\{\lambda_\nu\}}\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}dK(z)g(x)\\ &=\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}-\{\lambda_\nu\}}\frac{1}{\lambda^n}\prod_{p=1}^{n}\left\{\sum_{k=0}^{\infty}\left(\frac{\zeta_p z}{\lambda}\right)^k\right\}dK(z)g(x)\\ &=\frac{1}{\lambda^n}\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}-\{\lambda_\nu\}}\left\{1+\sum_{k=1}^{\infty}\frac{1}{\lambda^k}\kappa_k(\zeta_1, \zeta_2, \cdots, \zeta_n)z^k\right\}dK(z)g(x)\\ &=\frac{1}{\lambda^n}\left\{g(x)-\sum_{\nu=1}^{\infty}c_\nu\varphi_\nu(x)+\sum_{k=1}^{\infty}\frac{1}{\lambda^k}\kappa_k(\zeta_1, \zeta_2, \cdots, \zeta_n)c^k\sum_{\mu=1}^{\infty}(g, \psi_\mu)\sum_{j=1}^{\infty}\beta_{\mu j}^{(k)}\psi_j(x)\right\},\end{aligned}$$

where $\kappa_k(\zeta_1, \zeta_2, \cdots, \zeta_n)$ denotes the sum $\sum_{i,j,\dots,l=1}^{n}\zeta_i\zeta_j\cdots\zeta_l$ under the conditions $i\leqq j\leqq\cdots\leqq l$ and $\frac{i}{i}+\frac{j}{j}+\cdots+\frac{l}{l}=k$. Hence we have the desired solution (2) holding for almost every $x\in\Delta$. Furthermore, if we denote the subspaces determined by $\{\varphi_\nu\}$ and $\{\psi_\mu\}$ by $\mathfrak{M}$ and $\mathfrak{N}$ respectively and set $g=g_1+g_2$ where $g_1\in\mathfrak{M}$ and $g_2\in\mathfrak{N}$, then

$$\left\|\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x)\right\|^2\leqq(|\lambda|-\sigma)^{-2n}\|g_1\|^2\qquad(\sigma<|\lambda|<\infty)$$

and

$$\left\|\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}-\{\lambda_\nu\}}\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}dK(z)g(x)\right\|^2\leqq(|\lambda|-\sigma)^{-2n}\|g_2\|^2\qquad(\sigma<|\lambda|<\infty).$$

Consequently the right-hand side of (2) is of course an element of $L_2(\Delta, \rho)$.

We next consider the function $\chi(\lambda)=(f_\lambda, h)=\left(\prod_{p=1}^{n}(\lambda I-\zeta_p N_1)^{-1}g, h\right)$ $(\sigma<|\lambda|<\infty)$ defined in the statements of the present theorem. If we denote by Δ_c the continuous spectrum of N_1, then every point of the sets $\{\zeta_p\lambda_\nu\}_{\substack{p=1,2,\dots,n\\ \nu=1,2,3,\cdots}}$ and $\{\zeta_p\Delta_c\}_{p=1,2,\dots,n}$ is a singularity of

$\left(\prod_{p=1}^{n}(\lambda I-\zeta_p N_1)^{-1}g,\, h\right)$ for the domain $\{\lambda: |\lambda|<\infty\}$ but this function is regular elsewhere. Now, we have

$$\chi(\lambda)=\left(\int_{\{\lambda_\nu\}\cup\Delta_\sigma}\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}dK(z)g,\, h\right)$$

and here $\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}$ is decomposed by partial fractions. In addition, the ordinary part of $\chi(\lambda)$ is zero. Accordingly we can establish (A) by the same reasoning as that used to prove Theorem 30 in my previous paper [2] and can derive the expansion

$$\chi\left(\frac{r}{\kappa}e^{i\theta}\right)=\sum_{\nu=1}^{\infty}a_\nu(r)\left(\frac{\kappa}{e^{i\theta}}\right)^{\nu}$$

$$\left(a_\nu(r)=\frac{1}{\pi}\int_0^{2\pi}\chi(re^{it})\cos\nu t\, dt,\ \sigma<r<\infty,\ 0<\kappa<1\right)$$

from the same method as that applied to prove Theorem 36 in the same paper. By making use of this expansion, we can also establish (B) and (C), as will be found from the method of the proof of Theorem 43 in [3]. Furthermore (D) and (E) are shown by reasonings exactly like those applied to prove Theorem 48 in [4] and to prove Theorem 55 in [5], respectively.

Lastly we shall turn to the proof of (F).

By the definition of m_κ we can write

$$\begin{aligned} f_\lambda(x)&=\prod_{p=1}^{n}(\lambda I-\zeta_p N_1)^{-1}g(x) \qquad (\sigma<|\lambda|<\infty)\\ &=\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}}\sum_{\kappa=1}^{j}\sum_{\alpha=1}^{m_\kappa}A_\kappa^{(\alpha)}(z)(\lambda-\zeta_\kappa z)^{-\alpha}dK(z)g(x),\end{aligned}$$

where any $A_\kappa^{(\alpha)}(z)$ is a rational function of z such that $|A_\kappa^{(\alpha)}(z)|<\infty$ for $|z|\leqq \sup_\nu|\lambda_\nu|$. On the other hand, if we put

$$K=\max_{\alpha,\kappa}\{\sup_{|z|\leqq\sup_\nu|\lambda_\nu|}|A_\kappa^{(\alpha)}(z)|\}$$

then

$$\left|\left(\int_{\{z:|z|\leqq\sup_\nu|\lambda_\nu|\}}A_\kappa^{(\alpha)}(z)(\lambda-\zeta_\kappa z)^{-\alpha}dK(z)g,\, h\right)\right|\leqq K(r-\sigma_\kappa)^{-\alpha}\,||g||\,||h||$$

$$(\lambda=re^{i\theta},\ \sigma_\kappa=|\zeta_\kappa|\sup_\nu|\lambda_\nu|\leqq\sigma<r<\infty),$$

so that

$$\begin{aligned}\overset{+}{\log}|\chi(re^{-it})|&=\overset{+}{\log}|(f_{re^{-it}},h)|\\ &\leqq\sum_{\kappa=1}^{j}\sum_{\alpha=1}^{m_\kappa}\overset{+}{\log}[K(r-\sigma_\kappa)^{-\alpha}\,||g||\,||h||]+\log n\\ &\leqq\sum_{\kappa=1}^{j}\frac{m_\kappa(m_\kappa+1)}{2}\overset{+}{\log}[(r-\sigma)^{-1}]+n\overset{+}{\log}[K\,||g||\,||h||]+\log n.\end{aligned}$$

This final inequality permits us to conclude that

$$\varlimsup_{r\to\sigma+0}\frac{m(r,\infty)}{\log[(r-\sigma)^{-1}]}\leqq\sum_{k=1}^{j}\frac{m_\kappa(m_\kappa+1)}{2},$$

as we wished to prove.

Theorem 2. *Let all the notations defined in the statements of Theorem* 1 *be again used here, and let* $\sigma'=\dfrac{\sup_\nu|\lambda_\nu|}{\min_p\{|\zeta_p|\}}<\infty$. *Then the integral equation*

$$(3)\qquad \left(N_n+\sum_{p=1}^{n}a_p\lambda^p N_{n-p}\right)f(x)=g(x)\qquad (a_n\neq 0,\ N_0=I,\ \sigma'<|\lambda|<\infty)$$

has a uniquely determined solution

$$(4)\qquad \tilde{f}_\lambda(x)=\frac{1}{a_n}\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}\left(\lambda-\frac{\lambda_\nu}{\zeta_p}\right)^{-1}\varphi_\nu(x)+$$

$$\frac{1}{a_n\lambda^n}\left\{g(x)-\sum_{\nu=1}^{\infty}c_\nu\varphi_\nu(x)+\sum_{k=1}^{\infty}\frac{1}{\lambda^k}\kappa_k(\zeta_1^{-1},\zeta_2^{-1},\cdots,\zeta_n^{-1})c^k\sum_{\mu=1}^{\infty}(g,\psi_\mu)\sum_{j=1}^{\infty}\beta_{\mu j}^{(k)}\psi_j(x)\right\}.$$

If we set

$$\tilde{\chi}(\lambda)=(\tilde{f}_\lambda,h),\ \tilde{M}(r)=\max_{\theta\in[0,2\pi]}|\tilde{\chi}(re^{i\theta})|,\ \tilde{a}_\nu(r)=\frac{1}{\pi}\int_0^{2\pi}\tilde{\chi}(re^{it})\cos\nu t\,dt,$$

and

$$\tilde{m}(r,\infty)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|\tilde{\chi}(re^{-it})|\,dt$$

for any r *with* $\sigma'<r<\infty$, *then the results exactly analogous to* (A), (B), (C), (D), (E), *and* (F) *hold for* $\tilde{\chi}(x)$, $\tilde{M}(r)$, $\tilde{a}_\nu(r)$, $\tilde{m}(r,\infty)$, *and* σ'; *moreover if we denote by* Γ *a rectifiable closed Jordan curve containing the disc* $|\lambda|\leqq\max\{\sigma,\sigma'\}$ *inside itself, then*

$$(5)\qquad \chi(\lambda)\tilde{\chi}(\lambda)=\sum_{s=1}^{\infty}\left\{\frac{1}{2\pi i}\int_\Gamma\zeta^s\chi(\zeta)\tilde{\chi}(\zeta)d\zeta\right\}\frac{1}{\lambda^{s+1}}$$
$$(\max\{\sigma,\sigma'\}<|\lambda|<\infty),$$

where Γ *is positively oriented.*

Proof. Since, by the hypothesis on ζ_p, the roots of the equation $1+\sum_{p=1}^{n}a_p\lambda^p=0$ are given by $\zeta_p^{-1}(p=1,2,3,\cdots,n)$, it is easily verified that the given integral equation (3) is rewritten $a_n\prod_{p=1}^{n}\left(\lambda-\dfrac{N_1}{\zeta_p}\right)f(x)=g(x)$ and hence that (4) is the unique solution of (3). It is also clear that the results exactly analogous to (A), (B), (C), (D), (E), and (F) are valid for $\tilde{\chi}(\lambda)$, $\tilde{M}(r)$, $\tilde{a}_\nu(r)$, $\tilde{m}(r,\infty)$, and σ'. Suppose that Γ lies inside the circle $C\left\{\lambda:|\lambda|=\dfrac{r}{\kappa}\right\}$ with $\max\{\sigma,\sigma'\}<r<\infty$ and $0<\kappa<1$ and that C is positively oriented. Then, by Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_\Gamma\zeta^s\chi(\zeta)\tilde{\chi}(\zeta)d\zeta=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{\kappa}e^{i\theta}\right)^{s+1}\sum_{\nu=1}^{\infty}a_\nu(r)\left(\frac{\kappa}{e^{i\theta}}\right)^\nu\cdot\sum_{\nu=1}^{\infty}\tilde{a}_\nu(r)\left(\frac{\kappa}{e^{i\theta}}\right)^\nu d\theta$$
$$=r^{s+1}\{a_1(r)\tilde{a}_s(r)+a_2(r)\tilde{a}_{s-1}(r)+\cdots+a_s(r)\tilde{a}_1(r)\},$$

because of the fact that the expansions of $\chi\left(\dfrac{r}{\kappa}e^{i\theta}\right)$ and $\tilde{\chi}\left(\dfrac{r}{\kappa}e^{i\theta}\right)$

both are uniformly convergent with respect to θ. Since, on the other hand,

$$\chi(\lambda)\tilde{\chi}(\lambda)=\sum_{\nu=1}^{\infty}a_\nu(r)\left(\frac{r}{\lambda}\right)^\nu\cdot\sum_{\nu=1}^{\infty}\tilde{a}_\nu(r)\left(\frac{r}{\lambda}\right)^\nu \qquad \left(\lambda=\frac{r}{\kappa}e^{i\theta}\right)$$
$$=\sum_{s=1}^{\infty}\{a_1(r)\tilde{a}_s(r)+a_2(r)\tilde{a}_{s-1}(r)+\cdots+a_s(r)\tilde{a}_1(r)\}\left(\frac{r}{\lambda}\right)^{s+1},$$

we have the desired relation (5).

Remark 1. $\frac{1}{2\pi i}\int_\Gamma \frac{\chi(\zeta)\tilde{\chi}(\zeta)}{\zeta^s}d\zeta=0 \qquad (s=0, 1, 2, 3, \cdots).$

Remark 2. $\tilde{\chi}(\lambda)=\sum_{s=1}^{\infty}\left\{\frac{1}{2\pi i}\int_\Gamma \zeta^{s-1}\tilde{\chi}(\zeta)d\zeta\right\}\frac{1}{\lambda^s} \qquad (\sigma'<|\lambda|<\infty).$

Remark 3. The same result as that of Theorem 46 in [6] holds for the distribution of c'-points of $\chi(\lambda)$ in the domain $\{\lambda: \sigma<|\lambda|<\infty\}$, provided that σ is defined as in the statements of Theorem 1.

References

[1] S. Inoue: A note on the functional-representations of normal operators in Hilbert spaces. II. Proc. Japan Acad., **39**, 743-748 (1963).
——: Some applications of the functional-representations of normal operators in Hilbert spaces. XI, XII. Proc. Japan Acad., **40**, 391-395, 487-491 (1964).
[2] ——: Ditto. XIII. Ibid., **40**, 492-497 (1964).
[3] ——: Ditto. XVI. Ibid., **41**, 541-546 (1965).
[4] ——: Ditto. XVIII. Ibid., **41**, 911-914 (1965).
[5] ——: Ditto. XIX. Ibid., **41**, 915-918 (1965).
[6] ——: Ditto. XVII. Ibid., **41**, 702-705 (1965).