# 166. On Closed Mappings and M-Spaces. I

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**1. Introduction.** Recently K. Morita [1] has introduced the notion of $M$-spaces. We shall say that a topological space $X$ is an $M$-space if there exists a normal sequence $\{\mathfrak{U}_n \mid n=1, 2, \cdots\}$ of open coverings of $X$ which satisfies the condition below:

(*) If a family $\mathfrak{K}$ consisting of a countable number of subsets of $X$ has the finite intersection property and contains as a member a subset of $\mathrm{St}(x_0, \mathfrak{U}_n)$ for every $n$ and for some fixed point $x_0$ of $X$, then $\cap\{\bar{K} \in \mathfrak{K}\} \neq \phi$.

In this paper we shall introduce the notion of $M^*$-spaces which contains all $M$-spaces, and study some properties of these spaces. We shall say that a topological space $X$ is an $M^*$-space if there exists a sequence $\{\mathfrak{F}_n \mid n=1, 2, \cdots\}$ of locally finite closed coverings of $X$ which satisfies the condition (*). Of course we can assume without loss of generality that $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for every $n$. Theorems 2.3 and 2.4 will play the important roles in the proof of the main theorem which will be mentioned in the following paper "On closed mappings and $M$-spaces. II".

Finally the author wishes to express his hearty thanks to Prof. K. Morita who has given him valuable advices and encouragement.

**2. Some properties of M*-spaces.** **Lemma 2.1.** *Let $f$ be a closed continuous mapping of a $T_1$-space $X$ onto a topological space $Y$. If $f^{-1}(y)$ is countably compact for any point $y$ of $Y$, and if $\{F_\lambda \mid \lambda \in \Lambda\}$ is a locally finite collection of closed subsets of $X$, then $\{f(F_\lambda) \mid \lambda \in \Lambda\}$ is also a locally finite collection of closed subsets of $Y$.*

This lemma is due to A. Okuyama [4].

**Lemma 2.2.** *Let $X$ be an $M^*$-space with a sequence $\{\mathfrak{F}_n\}$ of locally finite closed coverings of $X$ such that $\{\mathfrak{F}_n\}$ satisfies the condition (*) and that $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for every $n$, and $C$ any countably compact subset of $X$, where $X$ is $T_1$. If $\mathfrak{K}$ is a family of countable number of subsets of $X$ which has the finite intersection property and contains as a member a subset of $St(C, \mathfrak{F}_n)$ for every $n$, then $\cap\{\bar{K} \mid K \in \mathfrak{K}\} \neq \phi$.*

**Proof.** First we note that, if $\mathfrak{F}$ is any locally finite closed covering of $X$, then a countably compact subset $C$ of $X$ intersects with only finite members of $\mathfrak{F}$. Hence for every $n$, $C$ intersects with only finite members of $\mathfrak{F}_n$. Consequently it is easy to see that for every $n$ there exists some element $F_n$ of $\mathfrak{F}_n$ such that $C\cap F_n\neq\phi$ and that $\mathfrak{K}\cap F_n(=\{K\cap F_n \mid K\in\mathfrak{K}\})$ has the finite intersection property. Let $x_n\in C\cap F_n$ for each $n$. Since $C$ is countably compact, we have $\bigcap\{\bar{A}_n\cap C \mid n=1, 2, \cdots\}\neq\phi$, where $A_n=\{x_n \mid k\geqq n\}$. Let $x_0\in\bigcap\{\bar{A}_n\cap C \mid n=1, 2, \cdots\}$. Then $x_0$ is an accumulation point of $\{x_n\}$. We shall prove that $\mathrm{St}(x_0, \mathfrak{F}_n)\cap\mathfrak{K}$ has the finite intersection property for every $n$. Indeed let us put

$$U_n(x_0)=X-\bigcup\{F \mid x_0\notin F,\ F\in\mathfrak{F}_n\}$$

for each $n$. Then $U_n(x_0)$ is open in $X$ and $U_n(x_0)\subset\mathrm{St}(x_0, \mathfrak{F}_n)$. Hence there exists some point $x_k$ such that $x_k\in U_n(x_0)$ and $k>n$. For this point $x_k$, we have

$$\mathrm{St}\,(x_k, \mathfrak{F}_k)\subset\mathrm{St}\,(x_k, \mathfrak{F}_n)\subset\mathrm{St}\,(x_0, \mathfrak{F}_n).$$

Since $x_k\in F_k$, $\mathfrak{K}\cap\mathrm{St}\,(x_k, \mathfrak{F}_k)$ has the finite intersection property, and hence $\mathfrak{K}\cap\mathrm{St}\,(x_0, \mathfrak{F}_n)$ has the same property. From this fact it follows that, if we put

$$\mathfrak{K}^*=\{K\cap\mathrm{St}\,(x_0, \mathfrak{F}_n) \mid K\in\mathfrak{K},\ n=1, 2, \cdots\},$$

then $\mathfrak{K}^*$ has the finite intersection property. Since $\mathfrak{K}^*$ contains as a member a subset of $\mathrm{St}\,(x_0, \mathfrak{F}_n)$ for every $n$, by our assumption we have $\bigcap\{\bar{K} \mid K\in\mathfrak{K}^*\}\neq\phi$, from which it follows that $\bigcap\{\bar{K} \mid K\in\mathfrak{K}\}\neq\phi$. This completes the proof.

**Theorem 2.3.** *Let $f$ be a closed continuous mapping of an $M^*$-space $X$ onto a topological space $Y$, where $X$ is $T_1$. If $f^{-1}(y)$ is countably compact for any point $y$ of $Y$, then $Y$ is also an $M^*$-space.*

**Proof.** Let $\{\mathfrak{F}_n\}$ be a sequence of locally finite closed coverings of $X$ such that $\{\mathfrak{F}_n\}$ satisfies the condition $(*)$ and that $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for every $n$. If we put $\mathfrak{F}_n=\{F_{n\lambda} \mid \lambda\in\Lambda_n\}$, $L_{n\lambda}=f(F_{n\lambda})$ and $\mathfrak{L}_n=\{L_{n\lambda} \mid \lambda\in\Lambda_n\}$, then by Lemma 2.1 $\mathfrak{L}_n$ is a locally finite closed covering of $Y$ for each $n$. Hence it is sufficient to prove that $\{\mathfrak{L}_n\}$ satisfies the condition $(*)$. Now let $\mathfrak{M}$ be a family of subsets of $Y$ which has the finite intersection property and contains as a member a subset of $\mathrm{St}\,(y_0, \mathfrak{L}_n)$ for every $n$ and for some fixed point $y_0$ of $Y$. Let us put further

$$\mathfrak{K}=\{f^{-1}(M) \mid M\in\mathfrak{M}\},$$

$$\mathfrak{K}_0=\{K\cap\mathrm{St}\,(f^{-1}(y_0), \mathfrak{F}_n) \mid K\in\mathfrak{K},\ n=1, 2, \cdots\}.$$

Clearly the family $\mathfrak{K}\cap\mathrm{St}\,(f^{-1}(y_0), \mathfrak{F}_n)$ has the finite intersection property, and hence $\mathfrak{K}_0$ has also the same property. Therefore by Lemma 2.2, we have $\bigcap\{\bar{A} \mid A\in\mathfrak{K}_0\}\neq\phi$, which implies that $\bigcap\{\overline{f^{-1}(M)} \mid M\in\mathfrak{M}\}\neq\phi$. Let $x\in\bigcap\{\overline{f^{-1}(M)} \mid M\in\mathfrak{M}\}$. Then it is clear that $f(x)\in\bigcap\{\bar{M} \mid M\in\mathfrak{M}\}$. Hence $\bigcap\{\bar{M} \mid M\in\mathfrak{M}\}\neq\phi$. Thus we complete the proof.

**Theorem 2.4.** *Let $f$ be a closed continuous mapping of a* *topological space $X$ onto an $M^*$-space $Y$. If $f^{-1}(y)$ is countably compact for any point $y$ of $Y$, then $X$ is also an $M^*$-space.*

**Proof.** Let $\{\mathfrak{L}_n\}$ be a sequence of locally finite closed coverings of $Y$ such that $\{\mathfrak{L}_n\}$ satisfies the condition (*) and that $\mathfrak{L}_{n+1}$ is a refinement of $\mathfrak{L}_n$ for every $n$. If we put $\mathfrak{L}_n=\{L_{n\lambda}\mid\lambda\in\Lambda_n\}$, $F_{n\lambda}=f^{-1}(L_{n\lambda})$, and $\mathfrak{F}_n=\{F_{n\lambda}\mid\lambda\in\Lambda_n\}$, then $\mathfrak{F}_n$ is obviously a locally finite closed coverings of $X$ and $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for every $n$. We shall prove that $\{\mathfrak{F}_n\}$ satisfies the condition (*). Now let $\mathfrak{K}=\{K_i\mid i=1,2,\cdots\}$ be a family of subsets of $X$ which has the finite intersection property and contains as a member a subset of $\mathrm{St}(x_0,\mathfrak{F}_n)$ for every $n$ and for some fixed point $x_0$ of $X$. We may assume without loss of generality that $K_i\cap K_j\in\mathfrak{K}$ for $i,j=1,2,\cdots$. Since $\{f(K_i)\mid i=1,2,\cdots\}$ has the finite intersection property and contains as a member a subset of $\mathrm{St}(f(x_0),\mathfrak{L}_n)$ for every $n$, we have $\cap\{\overline{f(K_i)}\mid i=1,2,\cdots\}\neq\phi$. Let $u_0\in\cap\{\overline{f(K_i)}\mid i=1,2,\cdots\}$ and put $C=f^{-1}(u_0)$. Then for any open subset $U$ of $X$ which contains $C$ and for any $K_i\in\mathfrak{K}$, we have $K_i\cap U\neq\phi$. Indeed, since $f\colon X\to Y$ is closed and onto, for such $U$ there exists an open subset $H$ of $Y$ such that $u_0\in H$ and $f^{-1}(H)\subset U$. Clearly $f(K_i)\cap H\neq\phi$ for every $i$, and hence $K_i\cap f^{-1}(H)\neq\phi$ for every $i$. Consequently we have $K_i\cap U\neq\phi$ for every $i$. From this fact, it can be shown that there exists a point $z_0$ of $C$ such that

$$z_0\in\cap\{\bar{K}_i\mid i=1,2,\cdots\}.$$

In fact, assume to be contrary. Then for any point $x$ of $C$, there exists a neighborhood $N(x)$ of $x$ such that $N(x)\cap K_i=\phi$ for some $K_i\in\mathfrak{K}$. Let us put

$$N_i=\cup\{N(x)\mid N(x)\cap K_i=\phi\}$$

for each $i$. Since $C$ is countably compact, $C$ is covered with finite members of $\{N_i\mid i=1,2,\cdots\}$, i.e.,

$$C\subset\cup\{N_{i(n)}\mid n=1,2,\cdots,k\}.$$

Since $N_{i(n)}\cap K_{i(n)}=\phi$ for each $n=1,2,\cdots,k$, we have

$$(\cup\{N_{i(n)}\mid n=1,2,\cdots,k\})\cap(\cap\{K_{i(n)}\mid n=1,2,\cdots,k\})=\phi,$$

where $\cap\{K_{i(n)}\mid n=1,2,\cdots,k\}\in\mathfrak{K}$. This is a contradiction. Thus we complete the proof.

Recently A. Okuyama [3] has proved the following theorem which is concerned with metrizability of $M$-spaces: In order that a topological space $X$ be metrizable it is necessary and sufficient that $X$ be a paracompact Hausdorff $M$-space and that the diagonal $\Delta$ of the product space $X\times X$ be a $G_\delta$-subset of $X\times X$.

As an analogous result, we obtain the following theorem which is concerned with metrizability of $M^*$-spaces.

**Theorem 2.5.** *In order that a topological space $X$ be metrizable it is necessary and sufficient that $X$ be a paracompact Hausdorff*

*$M^*$-space and that the diagonal $\Delta$ of the product space $X\times X$ be a $G_\delta$-subset of $X\times X$.*

Since Theorem 2.5 can be proved along the same line as in the proof of Okuyama [3], we omit the proof. But our proof is based on Morita's metrization theorem (cf. K. Morita [2]).

The problem whether a paracompact normal $M^*$-spaces is an $M$-space or not remains open.

As is easily seen, for an $M$-space $X$ there exists a sequence $\{\mathfrak{F}_n\}$ of locally finite closed coverings of $X$ such that

(a) $\{\mathfrak{F}_n\}$ satisfies the condition $(*)$,

(b) $\mathrm{St}\{\mathrm{St}(x,\mathfrak{F}_{n+1}),\mathfrak{F}_{n+1}\}\subset \mathrm{St}(x,\mathfrak{F}_n)$ for every $n$ and for any point $x$ of $X$.

Conversely we can prove the following

**Theorem 2.6.** *If $X$ is an $M^*$-space with a sequence $\{\mathfrak{F}_n\}$ of locally finite closed coverings of $X$ which satisfies the conditions* (a) *and* (b) *above, then $X$ is an $M$-space.*

**Proof.** Let us put $f(x,y)=0$ if $x\in \mathrm{St}(y,\mathfrak{F}_n)$ for all $n$; $f(x,y)=1$ if $x\notin \mathrm{St}(y,\mathfrak{F}_1)$; $f(x,y)=2^{-n}$ if $x\in \mathrm{St}(y,\mathfrak{F}_n)$ and $x\notin \mathrm{St}(y,\mathfrak{F}_{n+1})$. Then $f(x,y)$ satisfies the following conditions: (1) $f(x,x)=0$; (2) $f(x,y)=f(y,x)$; (3) for every $e>0$, $f(x,y)<e$ and $f(y,z)<e$ implies $f(x,z)<2e$. Hence by a theorem of Frink (cf. [5, p. 50]) there exists a pseudometric $r$ such that

$$1/4 f(x,y)\leqq r(x,y)\leqq f(x,y).$$

Let $i$ be an identity mapping of $X$ onto a pseudometric space $(X,r)$, and $g$ a quotient mapping of $(X,r)$ onto a metric space $T$, where $T$ is a quotient space obtained from $(X,r)$ by defining that two points $x$ and $y$ are equivalent if $y\in \mathrm{St}(x,\mathfrak{F}_n)$ for all $n$. Then the mapping $i$ is continuous. In fact, let $x$ be any point of $(X,r)$, $N_\varepsilon(x)$ an $\varepsilon$-neighborhood of $x$, i.e., $N_\varepsilon(x)=\{y\mid r(x,y)<\varepsilon\}$. Let $1/2^n<\varepsilon (n\geqq 1)$. Then $\mathrm{St}(x,\mathfrak{F}_n)=\{y\mid f(x,y)\leqq 1/2^n\}\subset N_\varepsilon(x)$, because $r(x,y)\leqq f(x,y)$. Since $\mathrm{Int}\,\mathrm{St}(x,\mathfrak{F}_n)\neq\phi$, the mapping $i: X\to (X,r)$ is continuous. Consequently, if we put $h=g\circ i$, the mapping $h: X\to T$ is also continuous. To prove closedness of $h$, let $A$ be a closed subset of $X$, and let $t_0\in\overline{h(A)}$, $x_0\in h^{-1}(t_0)$. Then we have $U_\varepsilon(t_0)\cap h(A)\neq\phi$ for any $\varepsilon>0$, where $U_\varepsilon(t_0)$ is an $\varepsilon$-neighborhood of $t_0$ in a metric space $T$. Hence it follows that $N_\varepsilon(x_0)\cap A\neq\phi$ for any $\varepsilon>0$. Let $\varepsilon<1/2^{n+2}$. Then we have $N_\varepsilon(x_0)\subset \mathrm{St}(x_0,\mathfrak{F}_n)$, because $N_\varepsilon(x_0)=\{y\mid r(x_0,y)<\varepsilon\}\subset\{y\mid r(x_0,y)<1/2^{n+2}\}\subset\{y\mid f(x,y)\leqq 1/2^n\}=\mathrm{St}(x_0,\mathfrak{F}_n)$. This shows that $\mathrm{St}(x_0,\mathfrak{F}_n)\cap A\neq\phi$ for every $n$. Consequently, from the condition (a) it follows that $\cap\{\mathrm{St}(x_0,\mathfrak{F}_n)\cap A\mid i=1,2,\cdots\}\neq\phi$. Let $x_1\in\cap\{\mathrm{St}(x_0,\mathfrak{F}_n)\cap A\mid n=1,2,\cdots\}$. Then we have $t_0=h(x_0)=h(x_1)\in h(A)$. Thus the mapping $h: X\to T$ is closed.

Let $\mathfrak{K}=\{K_i\mid i=1,2,\cdots\}$ be any family consisting of a countable number of subsets of $h^{-1}(t_0)$ $(t_0 \in T)$ and having the finite intersection property. Without loss of generality we may assume that $K_i \cap K_j \in \mathfrak{K}$ for any $i, j$. Let $x_0 \in h^{-1}(t_0)$. Then $K_i \subset \mathrm{St}(x_0, \mathfrak{F}_n)$ for all $n$. Hence we have $\cap\{\bar{K}_i \mid i=1,2,\cdots\} \neq \phi$ from (a). This shows that $h^{-1}(t_0)$ is countably compact. Hence by [1, Theorem 6.1] $X$ is an $M$-space. Thus we complete the proof.

In his paper [1], K. Morita has proved that $M$-spaces are $P$-spaces. Along the same line as in his direct proof of this result, we can prove that $M^*$-spaces are $P$-spaces, and further that $M^{**}$-spaces, which contain all $M^*$-spaces, are also $P$-spaces. We shall say that a topological space $X$ is an $M^{**}$-space if there exists a sequence $\{\mathfrak{A}_n\}$ of (not necessarily open or closed) coverings of $X$ such that

(1) $\{\mathfrak{A}_n\}$ satisfies the condition $(*)$,

(2) $\mathrm{Int}\,\mathrm{St}(x, \mathfrak{A}_n) \neq \phi$ for any point $x$ of $X$ and for every $n$.

Since a normal $P$-space is countably paracompact (cf. [1]), we obtain the following

**Theorem 2.7.** *Every $M^*$- and $M^{**}$-spaces are $P$-spaces, and hence every normal $M^*$- and $M^{**}$-spaces are countably paracompact.*

## References

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