100. *An Integral of the Denjoy Type. III*

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1. Introduction. This paper is concerned with the approximately continuous Denjoy integral (AD-integral) defined by the author [3]. The section 2 is devoted to simplify the theory of the AD-integral. The essential point is to use Romanovski's lemma ([2], p. 543). This idea was introduced by S. Izumi [2] who developed the theory of general Denjoy integral very simply using the lemma. In section 3, it will be proved that the AD-integral includes exactly the general Denjoy integral (D-integral) and the approximately continuous Perron integral (AP-integral) defined by J. C. Burkill [1].

2. The *AD*-integral. We begin by defining the notion of (ACG). A real valued function $f(x)$ defined on the closed interval $[a, b]$ is said to be (ACG) on the interval if $[a, b]$ is the sum of a countable number of *closed* sets on each of which $f(x)$ is absolutely continuous. Before introducing the AD-integral we need some preparations.

Lemma 1. *If a non-void closed set E is the sum of a countable number of closed sets E_k, then there exists an interval (l, m) containing points of E and an integer k such that $(l, m)\cdot E\subset E_k$.*

For the proof, see, for example, [5], p. 143.

Lemma 2 (**Romanovski**). *Let $\boldsymbol{F}$ be a system of open intervals in $I_0=(a, b)$ such that*

(i) *if $I_k\in\boldsymbol{F}$ $(k=1, 2, \cdots, n)$ and $\left(\bigcup_{k=1}^{n}\bar{I}_k\right)^{\circ}=I$ is an open interval then $I\in\boldsymbol{F}$.*

(ii) *$I\in\boldsymbol{F}$ and $I'\subset I$ imply $I'\in\boldsymbol{F}$.*

(iii) *if $\bar{I}'\subset I$ implies $I'\in\boldsymbol{F}$, then $I\in\boldsymbol{F}$.*

(iv) *if $\boldsymbol{F}_1$ is a subsystem of $\boldsymbol{F}$ such that $\boldsymbol{F}_1$ does not cover I_0, then there is an $I\in\boldsymbol{F}$ such that $\boldsymbol{F}_1$ does not cover I.*

Then $I_0\in\boldsymbol{F}$.

Lemma 3. *If $f(x)$ is absolutely continuous on $[a, b]$ and if $f'(x)=0$ a.e. then $f(x)$ is constant on $[a, b]$.*

Theorem 1. *If $f(x)$ is approximately continuous, (ACG) on $[a, b]$ and if $AD\ f(x)=0$ a.e. then $f(x)$ is constant on $[a, b]$.*

Proof. Let $\boldsymbol{F}$ be a system of all open intervals of (a, b) in which f is constant. $\boldsymbol{F}$ satisfies evidently the conditions (i), (ii), and (iii) in Lemma 2. If we show that (iv) is satisfied, then, by Lemma 2, (a, b) is contained in $\boldsymbol{F}$, and therefore by approximate continuity, f is constant on $[a, b]$.

Let $\boldsymbol{F}_1$ be a subsystem of $\boldsymbol{F}$ and E be the set of points not covered by $\boldsymbol{F}_1$. Then E is closed, and $f(x)$ is constant in each complementary interval of E with respect to $[a, b]$. Since f is (ACG) on $[a, b]$, the interval $[a, b]$ is the sum of a countable number of closed sets E_k, $[a, b]=\bigcup_{k=1}^{\infty} E_k$, on each of which f is absolutely continuous. It follows from Lemma 1 that there exists an interval (l, m) and a natural number k such that

$$(l, m)\cdot E\subset E_k.$$

Hence f is absolutely continuous on $[l, m]\cdot E$. Since f is constant in each complementary interval of $[l, m]\cdot E$ with respect to $[l, m]$, f is absolutely continuous on $[l, m]$ and therefore $AD\,f(x)=f'(x)=0$ a.e. By Lemma 2, f is constant in $[l, m]$. Hence $(l, m)\in \boldsymbol{F}$ but $(l, m)\,\bar{\in}\, \boldsymbol{F}_1$, for (l, m) contains points of E, which completes the proof.

For the sake of Theorem 1, we can define the approximately continuous Denjoy integral as follows.

Definition 1. A function $f(x)$ is said to be AD-integrable on $[a, b]$ if there exists a function $F(x)$ which is approximately continuous, (ACG) on $[a, b]$ and $AD\ F(x)=f(x)$ a.e. The function $F(x)$ is called an indefinite integral of $f(x)$ and the definite integral of $f(x)$ on $[a, b]$, denoted by $(AD)\int_a^b f(t)dt$, is defined as $F(b)-F(a)$.

Uniqueness of the definite integral follows from Theorem 1.

3. Relations between the *D*-integral, the *AP*-integral and the *AD*-integral.

A function $f(x)$ is termed ACG on $[a, b]$ if $f(x)$ is continuous on $[a, b]$ and if $[a, b]$ is the sum of a countable number of sets on each of which $f(x)$ is absolutely continuous.

Definition 2. The function $f(x)$ is said to be D-integrable on $[a, b]$ if there is a function $F(x)$ which is ACG on $[a, b]$ and $ADF(x)=f(x)$ a.e. We define $(D)\int_a^b f(t)dt=F(b)-F(a)$.

A function $U(x)$ is termed upper function of $f(x)$ in $[a, b]$ if the following conditions are satisfied:

(i) $U(a)=0$,

(ii) $U(x)$ is approximately continuous on $[a, b]$,

(iii) $AD\ U(x)>-\infty$ at each point of $[a, b]$,

(iv) $AD\ U(x)\geqq f(x)$ at each point of $[a, b]$.

There is a corresponding definition of lower function $L(x)$.

Definition 3 (Burkill [1]). If $f(x)$ has upper and lower functions in $[a, b]$ and

$$\inf_U U(b) = \sup_L L(b),$$

then $f(x)$ is termed AP-integrable on $[a, b]$. The common value of the two bounds is called the definite AP-integral of $f(x)$ and is denoted by $(AP)\int_a^b f(t)dt$.

Theorem 2. *If $f(x)$ is D-integrable on $[a, b]$, then $f(x)$ is also AD-integrable on $[a, b]$ and the integrals coincide each other. There exists a function which is AD-integrable but not D-integrable on some interval.*

Proof. Let $f(x)$ be a D-integrable function on $[a, b]$. Then there is a function $F(x)$ which is ACG on $[a, b]$ and $AD\ F(x) = f(x)$ a.e. Let $[a, b] = \bigcup_{k=1}^{\infty} E_k$ where F is absolutely continuous on each E_k. It follows from continuity of F([6], p. 224) that F is also absolutely continuous on $\bar{E}_k$. Hence F is (ACG) on $[a, b]$, and

$$(AD)\int_a^b f(t)dt = (D)\int_a^b f(t)dt.$$

Next we shall construct a function which is AD-integrable but not D-integrable on some interval.

Let $I_n = [2^{-n+1} - 2^{-2n}, 2^{-n+1}]$ $(n = 1, 2, \cdots)$ be a sequence of closed intervals on $[0, 1]$. If we put $E = \bigcup_{n=1}^{\infty} I_n$ then the set E has zero density at 0, since for $2^{-n+1} - 2^{-2n} \leqq h \leqq 2^{-n+1}$,

$$\frac{|E(0, h)|}{h} \leqq \frac{\sum_{k=n}^{\infty} 2^{-2k}}{2^{-n+1} - 2^{-2n}} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

For simplicity we set $I_n = [a_n, b_n]$. Let $\varphi_n(x)$ $(n = 1, 2, \cdots)$ be a sequence of functions defined on $[0, 1]$ as follows:

$$\begin{aligned} \varphi_n(x) &= \sin^2\left\{\frac{x - a_n}{b_n - a_n}\pi\right\} \qquad &&\text{for } x \in I_n = [a_n, b_n], \\ &= 0 &&\text{elsewhere.} \end{aligned}$$

Finally we define $F(x) = \sum_{n=1}^{\infty} \varphi_n(x)$. Then $F(x)$ is continuous on $[0, 1]$ except at $x = 0$ where $F(x)$ is approximately continuous, because

$$\lim F(x) = 0 = F(0) \qquad (x \longrightarrow 0, x \in E^c)$$

and the set E^c has unit density at 0.

Since $\varphi_n(x)$ is absolutely continuous on the closed interval I_n and is zero elsewhere, $F(x)$ is (ACG) and is ordinary differetiable everywhere except at 0. If we put on $[0, 1]$

$$\begin{aligned} f(x) &= F'(x) \qquad &&(x \neq 0), \\ &= 0 &&(x = 0), \end{aligned}$$

then it follows from Definition 2 that the function $f(x)$ is AD-

integrable on $[0, 1]$. But $f(x)$ is not D-integrable on $[0, 1]$. Suppose that $f(x)$ is D-integrable. Then, by Definition 3, there exists a function $G(x)$ ACG with $AD\ G(x)=f(x)$ a.e. Since $AD(F(x)-G(x))=0$ a.e. and since $F-G$ is approximately continuous and (ACG), it follows from Theorem 1 that $F-G$ is constant on $[0, 1]$. This contradicts to the fact that G is continuous at 0 but F is not so.

Theorem 3. *The AD-integral is more general than the AP-integral.*

Proof. It was proved by the author ([3], Theorem 2) that if $f(x)$ is AP-integrable on $[a, b]$ then $f(x)$ is also AD-integrable on $[a, b]$ and the integrals have same value.

G. Tolstoff ([7], 658) has given the function which is D-integrable on $[0, 1]$ but not AP-integrable. Since the AD-integral includes the D-integral by Theorem 2, the above function gives a required example AD-integrable but not AP-integrable. This completes the proof.

References

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