27. *Concerning Paracompact Spaces**)

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Tukey [8] investigated spaces with certain property of the refinements of their open covers, called fully normal spaces, while Dieudonné's generalized compact spaces are those having locally finite refinements for the open covers. Fully normal spaces and paracompact spaces were shown to be the same by A. H. Stone [7]. Compact Hausdorff spaces are characterized by one of the equivalent properties: (1) Every open cover of the space of power $\geq\aleph_0$ (the cardinal number of the set of all positive integers) has a subcover of power $<\aleph_0$. (2) Every net in the space has a cluster point, and (3) Every ultrafiler converges to a point. The question arises, how do these properties of compact spaces reappear in paracompact spaces? [4, p. 208]. Corson [1] gave a characterization of paracompact spaces analoguous to the property (3) by showing that a space is paracompact if and only if every Cauchy-like ultrafilter converges to a point. Whether there exist the analogies of the first two properties remains an open problem, that is, "Is there a cardinal number $\aleph$ associated to a space such that the space is paracompact if and only if every open cover of power $\geq\aleph$ has a subcover of power $<\aleph$?" and "Is there a class of nets with the property: the paracompactness of the space is equivalent to the existence of a cluster point of each net in the class?" Theorem 1 in this note will give affirmative answers to the questions.

Deudonné [2] showed that the cartesian product of a paracompact space and a compact space is paracompact and Michael's [5] sharpened result is that the compactness of one of the coordinate spaces can be replaced by σ-compactness. The general statement, relative to the paracompactness of the product of two paracompact spaces, has been ruled out by Sorgenfrey's counter-example [6]. We will show in Theorem 2 that the product of a paracompact space and a locally compact paracompact space is paracompact.

Definition. Let $\{x_\delta\,;\,\delta\in D\}$ be a net. The family of the cardinal numbers of all cofinal subsets of $\{x_\delta\}$ contains a smallest number $\aleph$ which is called the least cardinal number of $\{x_\delta\}$.

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Theorem 1. *If X is a uniform space with the family $V=\{V_\delta;\zeta\in\Delta\}$ of all neighborhoods of the diagonal as a uniformity and $\aleph$ is the least number of the net $\{V_\delta\}$, then the following statements are equivalent:*

(1) *X is paracompact.*

(2) *Each net in X with the least cardinal number $\geq\aleph$ has a cluster point.*

(3) *There is a subcover of power $<\aleph$ of each open cover of X with cardinal number $\geq\aleph$.*

Proof. (1) implies (2).

We assume without loss of generality that X has no closed subset of isolated points of power $\geq\aleph$. Let $S=\{x_\delta;\delta\in D\}$ be a net in X with the least cardinal number $\geq\aleph$. Suppose S has no cluster point. Let $A_\delta=\{x_{\delta'};\delta'\geq\delta, x_{\delta'}\in S\}$ and $F=\{\bar{A}_\delta\}$, where $\bar{A}_\delta$ is the closure of A_δ in X. The intersection of $\bar{A}_\delta$ for all $\delta\in D$ is void while the members of any subfamily of F contains a common nonvoid subset of X if the power of the subfamily is less than $\aleph$. Then $\mathfrak{G}=\{x-\bar{A}_\delta;\delta\in D\}$ is an open cover of X containing no subcover of power $<\aleph$. Since X is paracompact, $\mathfrak{G}$ has a locally finite refinement $\mathfrak{B}$. Choosing a point t from each member of $\mathfrak{B}$, we have a closed discrete set of non-isolated points $K=\{t_\nu\}$ whose cardinal number is $\geq\aleph$. Let W be a cofinal subset of V with $\aleph$ as its least cardinal number and let $K_\delta=K-\{t_\delta\}$ for some δ.

There is a one-to-one correspondence between the members V_δ of W and the points t_δ of a subset K_1 of K such that each t_δ has an open neighborhood Q_δ with the property: $V_\delta(t_\delta)\not\subset Q_\delta$ and $Q_\delta\cap K_\delta=\phi$. The family consisting of all Q_δ and $X-K_1$ forms an open cover of X that is not even. This is a contradiction.

(2) implies (3).

Let $\mathfrak{A}$ be an open cover of x of power $\geq\aleph$ which has no subcover of power $<\aleph$. The family of complements of the members of $\mathfrak{A}$ has finite intersection property. Pick a point x_δ from the complement of each member A_δ of $\mathfrak{A}$. The net $\{x_\delta\}$ with the least cardinal number $\geq\aleph$ has a cluster point x_0 belonging to the void set $\bigcap_{A_\delta\in\mathfrak{A}}\complement(A_\delta)$ and a contradiction is reached.

(3) implies (1).

Suppose that X is not paracompact. There exists an uneven open cover $\mathfrak{A}$ of X. To each $V_\delta\in V$ a point x_δ can be so chosen that $V_\delta(x_\delta)$ is not contained in any member of $\mathfrak{A}$. If the net $\{x_\delta\}$ has a cluster point belonging to some member P of $\mathfrak{A}$, we can find $V_{\delta'}, V_{\delta''}$ such that $V_{\delta'}(x_0)\subset P$ and $V^2_{\delta''}\subset V_{\delta'}$. Let $x_{\bar{\delta}}$ be a point in $V_{\delta''}(x_0)$ and $V_{\delta'}\subset V_{\delta''}$. Then $V_{\bar{\delta}}(x_{\bar{\delta}})\subset V_{\delta''}(x_{\bar{\delta}})\subset V^2_{\delta''}(x_0)\subset V_{\delta'}(x_0)\subset P$

which is impossible.

Let A_{δ_0} be the set of all x_δ for $\delta \geq \delta_0$. The intersection of closures $\bar{A}_\delta$ for all δ is void since $\{x_\delta\}$ has no cluster point. The complements of $\bar{A}_\delta$ for all δ form an open cover of X that has $\aleph$ as its least cardinal number and has no subcover of power $<\aleph$.

Theorem 2. *The product of a paracompact space and a locally compact paracompact space is paracompact.*

Proof. Let X_1 be a paracompact space and X_2 a locally compact paracompact space. Let $\mathfrak{A}$ be an open cover of $X_1 \times X_2$. To each $(x_1, x_2) \in X_1 \times X_2$ there are an open neighborhood $N_1(x_1, x_2)$ of x_1 in X_1 and an open neighborhood $N_2(x_1, x_2)$ of x_2 in X_2 such that $N_1(x_1, x_2) \times N_2(x_1, x_2)$ lies in a member of $\mathfrak{A}$. Then $\{N_1(x_1, x_2) \times N_2(x_1, x_2)\,;\, x_2 \in X_2\}$ is an open cover of $\{x_1\} \times X_2$. Denote a locally finite refinement of the open cover $\{N_2(x_1, x_2)\,;\, x_2 \in X_2\}$ of X_2 by $R_2(x_1)$. Each $x_2 \in X_2$ has an open neighborhood $S(x_2)$ intersecting every locally finite open cover of X_2 in finitely many members. If $\mathfrak{G} = \{G_\alpha\,;\, \alpha \in \Delta\}$ is a locally finite refinement of $\{S(x_2)\,;\, x_2 \in X_2\}$, each G_α meets a finite number of the members of $R_2(x_1)$ contained in $N_2(x_1, x_{2_i})$, $i=1, \cdots, n$. Let $W(x_1, \alpha) = \bigcap_{i=1}^{n} N_1(x_1, x_{2_i})$. The open cover $\{W(x_1, \alpha)\,;\, x_1 \in X_1\}$ has a locally finite refinement $R_1(\alpha)$. To each $A_1 \in R_1(\alpha)$, a point $x_1 \in X_1$ is so chosen that $A_1 \subset W(x_1, \alpha)$, and denote this set of points in X_1 by $P_1(\alpha)$.

Let $\mathfrak{B} = \{A_1 \times A_2\,;\, A_1 \subset W(x_1, \alpha),\, A_2 \in R_1(x_1)$ for $A_1 \in R_1(\alpha),\, x_1 \in P_1(\alpha)$, and $\alpha \in \Delta\}$.

We show first that $\mathfrak{B}$ is a refinement of $\mathfrak{A}$. For each $(a_1, a_2) \in X_1 \times X_2$ there are $G_\alpha \in \mathfrak{G}$ and $A_1 \in R_1(\alpha)$ such that $a_1 \in A_1$ and $a_2 \in G_\alpha$. Then $A_1 \subset W(x_1, \alpha)$ for some $x_1 \in P_1(\alpha)$ and we can find $A_2 \in R_1(x_1)$ with $a_2 \in A_2$. It is clear that $A_1 \times A_2$ is contained in some member of $\mathfrak{A}$ and $\mathfrak{B}$ is a refinement of $\mathfrak{A}$.

In order to show the local finiteness of $\mathfrak{B}$, let O_2 be an open neighborhood of a_2 meeting $G_{\alpha_1}, \cdots, G_{\alpha_m}$ and let O_{1_i} be an open neighborhood of a_1 meeting the members $A_{\alpha_i 1}, \cdots, A_{\alpha_i h_i}$ of $R_1(\alpha_i)$ for $i=1, \cdots, m$. The neighborhood $O_1 \times O_2$ of (a_1, a_2), where $O_1 = \bigcap_{i=1}^{m} O_{1_i}$, intersects only finitely many members of $\mathfrak{B}$, because G_{α_i} and hence O_2 intersect a finite number of the members of $R_1(x_{\alpha_i 1}), \cdots, R_1(x_{\alpha_i h_i})$ for $i=1, \cdots, m$.

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