205. *On Maharam Subfactors of Finite Factors. II*

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(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1967)

1. In [1], we have showed that there exists a Maharam II_1-subfactor in a II_1-factor and that every I_n-subfactor of a II_1-factor is a Maharam subfactor. In this paper, as a continuation of [1], we shall show that there exists a non-Maharam proper II_1-subfactor in a II_1-factor.

2. Let $\mathcal{A}$ be a II_1-factor acting on a Hilbert space $\mathfrak{H}$ and $\mathcal{B}$ the full operator algebra on the 2-dimensional Hilbert space $\mathfrak{K}$. Let (T_{ij}), $i, j = 1, 2$, $T_{ij} \in \mathcal{A}$, be the matrix representation of an operator T of the tensor product $\mathcal{A}\otimes\mathcal{B}$, and φ the faithful normal trace of $\mathcal{A}$ with $\varphi(1)=1$. Then the functional ϕ on $\mathcal{A}\otimes\mathcal{B}$ defined by

$$\phi(T)=\frac{[\varphi(T_{11})+\varphi(T_{22})]}{2} \qquad \text{for } T=(T_{ij})\in\mathcal{A}\otimes\mathcal{B}$$

is a faithful normal trace on $\mathcal{A}\otimes\mathcal{B}$ and satisfies the equality $\phi(1)=1$.

For $T=(T_{ij})\in\mathcal{A}\otimes\mathcal{B}$, let

$$T^{\varepsilon}=\left(\frac{\delta_{ij}(T_{11}+T_{22})}{2}\right).$$

Then the mapping $\mathcal{A}\otimes\mathcal{B}\ni T\rightarrow T^{\varepsilon}\in\mathcal{A}\otimes C_{\mathfrak{K}}$ satisfies the following properties: For any complex numbers α and β, and any S and T of $\mathcal{A}\otimes\mathcal{B}$,

(1) $$(\alpha S+\beta T)^{\varepsilon}=\alpha S^{\varepsilon}+\beta T^{\varepsilon},$$

(2) $$T^{*\varepsilon}=T^{\varepsilon *},$$

(3) $$(S^{\varepsilon}T)^{\varepsilon}=(ST^{\varepsilon})^{\varepsilon}=S^{\varepsilon}T^{\varepsilon},$$

(4) $$\phi(T^{\varepsilon})=\phi(T),$$

(5) $$(\mathcal{A}\otimes\mathcal{B})^{\varepsilon}=\{T^{\varepsilon};\ T\in\mathcal{A}\otimes\mathcal{B}\}=\mathcal{A}\otimes C_{\mathfrak{K}},$$

(6) $$1^{\varepsilon}=1.$$

(1), (2), (5), and (6) are obvious. To prove (3), let $S=(S_{ij})$ and $T=(T_{ij})$, where $T_{ij}, S_{ij}\in\mathcal{A}$ for $i, j=1, 2$. Then

$$S^{\varepsilon}=\left(\frac{\delta_{ij}(S_{11}+S_{22})}{2}\right) \quad \text{and} \quad T^{\varepsilon}=\left(\frac{\delta_{ij}(T_{11}+T_{22})}{2}\right).$$

Hence we have

$$S^{\varepsilon}T=\left(\sum_{j=1}^{2}\delta_{ij}\frac{S_{11}+S_{22}}{2}T_{jk}\right)$$
$$=\left(\frac{S_{11}+S_{22}}{2}T_{ik}\right).$$

Therefore

$$(S^{\varepsilon}T)^{\varepsilon}=\left(\frac{1}{2}\delta_{ij}\left[\frac{1}{2}(S_{11}+S_{22})(T_{11}+T_{22})\right]\right)$$
$$=\frac{1}{4}(\delta_{ij}(S_{11}+S_{22})(T_{11}+T_{22}))$$
$$=S^{\varepsilon}T^{\varepsilon}\ .$$

Similarly, we have $(ST^{\varepsilon})^{\varepsilon}=S^{\varepsilon}T^{\varepsilon}$. Hence (3) is satisfied. For (4), we have

$$\phi(T^{\varepsilon})=\frac{1}{2}\phi(\delta_{ij}(T_{11}+T_{22}))$$
$$=\frac{1}{2}\left[\frac{1}{2}(\varphi(T_{11}+T_{22})+\varphi(T_{11}+T_{22}))\right]$$
$$=\frac{1}{2}\varphi(T_{11}+T_{22})=\phi(T).$$

Therefore, ε is the conditional expectation of $\mathcal{A}\otimes\mathcal{B}$ relative to $\mathcal{A}\otimes C_{\mathfrak{K}}$ in the sense of Umegaki [3].

If there exists a projection $E=(E_{ij})$, $i, j=1, 2$, $E_{ij}\in\mathcal{A}$, in $\mathcal{A}\otimes\mathcal{B}$ such as $E^{\varepsilon}=1/5$, then we have the following equalities:

a) $$\frac{1}{2}(E_{11}+E_{22})=\frac{1}{5},$$

b) $$E_{11}{}^{*}=E_{11},\quad E_{22}{}^{*}=E_{22},\quad \text{and}\quad E_{12}{}^{*}=E_{21},$$

c) $$E_{11}E_{11}{}^{*}+E_{12}E_{21}=E_{11},$$

d) $$E_{21}E_{12}+E_{22}E_{22}{}^{*}=E_{22},$$

e) $$E_{11}E_{12}+E_{12}E_{22}=E_{12}.$$

By a) and e), we have

f) $$\frac{3}{5}E_{12}=E_{11}E_{12}-E_{12}E_{11}.$$

By a) and f), we have

$$3E_{22}E_{12}=-(E_{11}E_{12}+2E_{12}E_{11}),$$

then we have

g) $$3E_{22}E_{12}E_{12}{}^{*}=-(E_{11}E_{12}E_{12}{}^{*}+2E_{12}E_{11}E_{12}{}^{*}).$$

E_{22}, $E_{12}E_{12}{}^{*}$, and E_{11} are mutually commutative by a) and c), whence the left side of g) is nonnegative and the right side of g) is nonpositive, and so we have

h) $$E_{22}E_{12}E_{12}{}^{*}=0.$$

By c) and h),

i) $$E_{22}E_{11}(1-E_{11})=0.$$

On the other hand, by a), c), and d),

$$0\leq E_{11},\ E_{22}\leq\frac{2}{5}$$

and

$$E_{11}E_{22}=E_{22}E_{11}.$$

Therefore, applying *i*), we have

j) $$E_{11}E_{22}=E_{22}E_{11}=0.$$

Let $F=5E_{11}/2$, then F is a projection in $\mathcal{A}$ by a), b), and j). However, we shall show, in the below, that this projection F is 0. By the assumption,

$$E_{11}=\frac{2}{5}F, \quad E_{22}=\frac{2}{5}(1-F)$$

and

$$E_{12}E_{12}^*=\frac{6}{25}F,$$

and by f), we have

$$\frac{3}{5}\frac{6}{25}F=\frac{2}{5}F\frac{6}{25}F-E_{12}E_{11}E_{12}^*,$$

therefore, we have

$$0\leq\frac{1}{5}\frac{6}{25}F=-E_{12}E_{11}E_{12}^*\leq 0.$$

Thus, we have

$$E_{11}=0,\ E_{12}=0, \quad \text{and} \quad E_{22}=\frac{2}{5}.$$

Applying d), we have finally

$$\left(\frac{2}{5}\right)^2=\frac{2}{5},$$

which is a contradiction.

Hence $\mathcal{A}\otimes C_{\mathfrak{K}}$ is not a Maharam subfactor of $\mathcal{A}\otimes\mathcal{B}$, whence we have proved.

Theorem 1. *Let $\mathcal{A}$ be a II_1-factor acting on a Hilbert space $\mathfrak{H}$ and $\mathcal{B}$ the full operator algebra on a 2-dimensional Hilbert space $\mathfrak{K}$, then $\mathcal{A}\otimes C_{\mathfrak{K}}$ is not a Maharam subfactor of $\mathcal{A}\otimes\mathcal{B}$.*

Theorem 1 gives an example of a II_1-factor whichh as a non-Maharam proper II_1-subfactor. The following theorem has more general character:

Theorem 2. *Let $\mathcal{A}$ be a II_1-factor. Then there exists a proper II_1-subfactor $\mathcal{B}$ of $\mathcal{A}$ which is not a Maharam subfactor.*

Proof. Let C be a I_2-subfactor of $\mathcal{A}$. Then there exists a II_1-factor $\mathcal{B}$ such that $\mathcal{B}\otimes C$ is isomorphic to $\mathcal{A}$ by a lemma of Misonou [2]. Being considered $\mathcal{B}$ as a subfactor of $\mathcal{A}$, $\mathcal{B}$ is not a Maharam subfactor of $\mathcal{A}$ by Theorem 1. Clearly, $\mathcal{B}$ is a proper subfactor. Hence Theorem 2 is established.

3. In this oppotunity, we wish to give a correction on the preceding [1; Lemma 1]: In our proof, it is necessary to assume that $\mathcal{A}\cap\mathcal{A}_1$ and $\mathcal{B}\cap\mathcal{B}_1$ are semi-finite. Namely, the corrected statement of [1; Lemma 1] is as following: Let $\mathcal{A}$ and $\mathcal{A}_1$ (resp. $\mathcal{B}$ and $\mathcal{B}_1$) be semi-finite von Neumann algebras acting on a Hilbert space $\mathfrak{H}$ (resp. $\mathfrak{K}$). If $\mathcal{A}\cap\mathcal{A}_1$ and $\mathcal{B}\cap\mathcal{B}_1$ are semi-finite, then we have

$$(*)\qquad (\mathcal{A}\otimes\mathcal{B})\cap(\mathcal{A}_1\otimes\mathcal{B}_1)=(\mathcal{A}\cap\mathcal{A}_1)\otimes(\mathcal{B}\cap\mathcal{B}_1).$$

It seems to the author that the semi-finiteness assumption of the lemma is superfluous since (*) is able to prove for any von Neumann algebras using a theorem in an unpublished paper of M. Tomita.

References

[1] H. Choda: On Maharam subfactors of finite factors. Proc. Japan Acad., **43**, 451-455 (1967).

[2] Y. Misonou: On divisors of factors. Tohoku Math. J., **8**, 63-69 (1956).

[3] H. Umegaki: Conditional expectation in an operator algebra. Tohoku Math. J., **6**, 177-181 (1954).