212. A Note on Products of Spaces with Generalized Compactness Properties*)

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Let m and n be infinite cardinals, $m \geq n$, and let C_n (respectively C_n^m) be the collection of all topological spaces Y with the property that every open cover C of Y (with card $C \leq m$) has an open refinement $\mathcal{R}$ with card $\mathcal{R} < n$. L. H. Martin [6] has shown that if X is compact, if Y belongs to C_n or C_n^m, and if $m = n$, then $X \times Y$ belongs to C_n or C_n^m, respectively. The purpose of this paper is to extend Martin's result to the case that $m \neq n$.

The following characterization of C_n and C_n^m was suggested by a definition of I. S. Gál [3]. It is clear that

Lemma 1. *C_n (respectively C_n^m) is the collection of all topological spaces Y such that, if $\{F_a : a \in \Lambda\}$ is a family of closed sets in Y (of cardinality $\leq m$) with the property that any subcollection of $\{F_a : a \in \Lambda\}$ with cardinality $< n$ has a nonempty intersection, then $\bigcap \{F_a : a \in \Lambda\} \neq \phi$.*

As special cases of C_n^m where $m \neq n$, note that

(i) $C_{\aleph_0}^m$ is the collection of all m-compact spaces (in the sense of Frolík [1]).

(ii) $C_{m'}^n$ is the collection of all (m, n)-compact spaces**) (in the sense of Gál [3]).

Generalizing a result of Z. Frolík [2] gives

Lemma 2. *Let f be a closed map from a space P into a space Y. If $Y \in C_n$ (respectively C_n^m) and $f^{-1}(y) \in C_n(C_n^m)$ for each $y \in Y$, then $P \in C_n(C_n^m)$.*

Proof. Suppose $\{F_a : a \in \Lambda\}$ is a family of closed subsets of P (with cardinality $\leq m$) such that any subcollection of $\{F_a : a \in \Lambda\}$ with cardinality $< n$ has a nonempty intersection. Without loss of generality we may assume that if $\Gamma \subset \Lambda$ and card $\Gamma < n$, then $\bigcap \{F_a : a \in \Gamma\}$ belongs to $\{F_a : a \in \Lambda\}$. Choose $y \in \bigcap \{f(F_a) : a \in \Lambda\}$. The space $E = f^{-1}(y)$ belongs to $C_n(C_n^m)$, and $\{F_a \cap E : a \in \Lambda\}$ is a family of closed subsets of E with the desired intersection property (and cardinality). Thus, by Lemma 1, $\bigcap \{F_a \cap E : a \in \Lambda\} \neq \phi$, and so $\bigcap \{F_a : a \in \Lambda\} \neq \phi$.

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**) Here m' is the least cardinal greater than m.

The following lemma was presented by Frolík [2].

Lemma 3. *Let X be a compact space and Y be a space. The projection map from $X\times Y$ onto Y is a closed map.*

As a consequence of Lemmas 2 and 3 we have

Theorem. *If X is a compact topological space and Y belongs to C_n or C_n^m, then $X\times Y$ belongs to C_n or C_n^m, respectively.*

Note that the proof of the theorem relies heavily on a characterization of generalized compactness properties in terms of closed sets. If $m=n$, Martin's product theorem also handles the collections P_n, P_n^m, M_n, and M_n^m, where $Y\in P_n(P_n^m)$ if and only if every open cover of Y (of cardinality $\leq m$) has a locally-finite refinement of cardinality $<n$, and $Y\in M_n(M_n^m)$ if and only if every open cover of Y (of cardinality $\leq m$) has a point-finite refinement of cardinality $<n$. Y. Hayashi [5] gives characterizations for P_n^m and M_n^m, for $m=n$, in terms of closed sets, and these characterizations lend themselves very nicely to the techniques used above; this, however, does not extend Martin's work. For the P and M spaces, in the case that $m\neq n$, the author still does not know if the general theorem is true.

References

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