158. *On the Existence of Discontinuous Solutions of the Cauchy Problem for Quasi-Linear First-Order Equations*

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1. **Introduction.** In recent years, interest in discontinuous solutions of the Cauchy problem for nonlinear partial differential equations has conciderably increased and much progress has been made for quasi-linear first-order equations of conservation type in a single space variable (see Oleinik [3] for a survey of literatures).

In the case of several space variables, using a finite difference scheme, Conway and Smoller [1] has proved the existence of weak solutions of the Cauchy problem

$$(1.1)\qquad u_t+\sum_{i=1}^{n}\frac{\partial f^i(u)}{\partial x_i}=0$$

with a bounded measurable initial function having locally bounded variation in the sense of Tonelli-Cesari. A function f is said to have locally bounded variation in the sense of Tonelli-Cesari over R^n if for any compact set K in R^n there exists a set N of measure zero such that

$$V^i(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)=\operatorname*{Var}_{K-N} f(x_1,\cdots,x_{i-1},\cdot,x_{i+1},\cdots,x_n),\ i=1,\cdots,n$$

is measurable and summable, and we denote by F the class of these functions.

The purpose of this paper is to prove the existence of weak solutions of the Cauchy problem of the type:

$$(1.2)\qquad u_t+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f^i(t,x,u)+g(t,x,u)=0,$$

$$(1.3)\qquad u(0,x)=u_0(x)\in F.$$

For simplicity, we restrict ourselves to the case $n=2$. But it will be easily seen that one can extend at once everything which we do in this case to the case $n\geqq 3$. Thus we shall consider the Cauchy problem

$$(1.4)\qquad u_t=\frac{\partial}{\partial x}f(t,x,y,u)+\frac{\partial}{\partial y}g(t,x,y,u)+h(t,x,y,u)=0,$$

$$(1.5)\qquad u(0,x,y)=u_0(x,y)\in F,$$

in the region

$$G=\{(t,x,y);\ 0\leqq t\leqq T<\infty,\ -\infty<x,y<\infty\}.$$

We call a function $u(t,x,y)$ a weak solution of (1.4), (1.5) if it satisfy the relation

$$(1.6)\quad \iiint_G [u\varphi_t+f\varphi_x+g\varphi_y-h\varphi]dxdydt+\iint_{t=0} u_0(x, y)\varphi(0, x, y)dxdy=0$$

for any C^1 function $\varphi=\varphi(t, x, y)$, equal to zero outside a finite region, also for $t=T$.

The assumptions concerning the f, g, and h are followings:

i) the f, g, and h, and also the partial derivatives $f_x, f_y, f_u, f_{xx}, f_{xy}, f_{yy}, f_{xu}, f_{yu}, g_x, g_y, g_u, g_{xx}, g_{xy}, g_{yy}, g_{xu}, g_{yu}, h_x, h_y,$ and h_u are continuous for all u and (t, x, y) in G, and bounded for bounded u and (t, x, y) in G;

ii) there exist continuously differentiable functions $V^1(v)$ and $V^2(v)$, defined for $v\geqq 0$, such that

$$\max_{\substack{(t,x,y)\in G\\ |u|\leqq v}} \left|f_x+\frac{1}{2}h\right| \leqq V^1(v), \qquad \frac{dV^1(v)}{dv}\geqq 0,$$

$$\max_{\substack{(t,x,y)\in G\\ |u|\leqq v}} \left|g_y+\frac{1}{2}h\right| \leqq V^2(v), \qquad \frac{dV^2(v)}{dv}\geqq 0,\text{[1]}$$

and such that for any $v_0\geqq 0$

$$(1.7)\qquad \int_{v_0}^{\infty}\frac{dv}{V^1(v)+V^2(v)}=\infty.$$

In other words, the results obtained here is

Theorem. *Let the f, g, and h satisfy the conditions* i) *and* ii). *If $u_0\in F$, then there exists a weak solution $u(t, x, y)$ of* (1.4), (1.5) *in G such that $u(t, x, y)$ is of locally bounded variation in the sense of Tonelli-Cesari in G, and $u(t, x, y)\in F$ for each fixed t, $0\leqq t\leqq T$.*

This theorem will be proved by means of finite difference scheme. The finite difference scheme used here is a slight modification of that of Conway and Smoller [1] and more closely related to that of Oleinik [3].

In section 2, we introduce the finite difference scheme and obtain estimates for the solution of these equations, corresponding to Lemmas 1, 2, 4 in Conway and Smoller [1]. Therefore, we can prove the theorem in the same way as in section 3 of [1]. In section 3, we shall prove the theorem. Section 4 consists of some remarks. Detailed proof will be published elsewhere.

2. Estimates for the difference equations. Let the domain G be covered by a grid defined by the planes

$$t=kr,\ x=mp,\ y=nq,$$

where r, p, and q are fixed positive numbers, k are integers such that $0\leqq k\leqq [T/t]$, and m and n assume all integers.

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1) In the case $n=m$, this condition becomes $\max\left|f^i_{x_i}+\frac{1}{m}g\right|\leqq V^i(v)$, $\frac{dV^i(v)}{dv}\geqq 0$, $i=1, \cdots, m$.

In G, we consider the finite difference scheme defined by

$$
\begin{aligned}
(2.1)\qquad & \frac{1}{r}\Big[u^{k+1}_{m,n}-\frac{1}{4}(u^k_{m-1,n-1}+u^k_{m-1,n+1}+u^k_{m+1,n-1}+u^k_{m+1,n+1})\Big] \\
& +\frac{1}{4p}[f^k_{m+1,n+1}+f^k_{m+1,n-1}-f^k_{m-1,n+1}-f^k_{m-1,n-1}] \\
& +\frac{1}{4q}[g^k_{m+1,n+1}+g^k_{m-1,n+1}-g^k_{m+1,n-1}-g^k_{m-1,n-1}] \\
& +\frac{1}{4}[2h^k_{m+1,n+1}+h^k_{m+1,n-1}+h^k_{m-1,n+1}]=0,
\end{aligned}
$$

where we are using notations

$t^k=kr,\ x_m=mp,\ y_n=nq,\ u^k_{m,n}=u(t^k,\ x_m,\ y_n),\ f^k_{m,n}=f(t^k,\ x_m,\ y_n,\ u^k_{m,n})$ etc.

Let us divide the grid points into four classes as follows:

$S_1=\{(t^k,\ x_m,\ y_n);$ both $k-m$ and $k-n$ are even$\}$,
$S_2=\{(t^k,\ x_m,\ y_n);$ $k-m$ is even and $k-n$ is odd$\}$,
$S_3=\{(t^k,\ x_m,\ y_n);$ $k-m$ is odd and $k-n$ is even$\}$,
$S_4=\{(t^k,\ x_m,\ y_n);$ both $k-m$ and $k-n$ are odd$\}$.

Then, by virtue of the obvious property of the finite difference scheme (2.1), it is easy to see that the values $u^k_{m,n}$ at the points of S_i and S_j for $i\neq j$ are computed independently. Hence, it is sufficient to consider $u^k_{m,n}$ only at the points of S_1.

It follows from (1.7) that for any $M^0, \alpha>0$ there exists a constant $M>0$ such that

$$(2.2)\qquad \int_{M^0}^{M}\frac{dv}{V^1(v)+V^2(v)+\alpha}\geqq T.\text{[2)]}$$

Lemma 1. *Let* $|u^0_{m,n}|\leqq M^0$ *for all* m *and* n, *and* A *and* B *be defined by*

$$A=\max_{\Omega}|f_u|,\qquad B=\max_{\Omega}|g_u|,$$

where $\Omega=\{(t,\ x,\ y,\ u);\ (t,\ x,\ y)\in G,\ |u|\leqq M\}$.

Then, if the stability requirements $Ar/p+Br/q<1$ *are fulfilled for sufficiently small* p *and* q, *we have* $|u^k_{m,n}|\leqq M$ *for all values of* k, m, *and* n.

If we let p and q so small that $p\cdot\max_{\Omega}|f_{xx}|+q\cdot\max_{\Omega}|g_{yy}|<\alpha$, then we obtain this lemma in an analogous way to Theorem 5.1 of Douglis [2].

In what follows we shall assume the stability condition $Ar/p+Br/q<1$, and let $u^k_{m,n}$ be solutions of the finite difference equation (2.1) with $|u^0_{m,n}|\leqq M^0$.

Put

$$(2.3)\qquad w^k_{m,n}=\frac{u^k_{m,n}-u^k_{m-2,n}}{2p},\qquad z^k_{m,n}=\frac{u^k_{m,n}-u^k_{m,n-2}}{2q},$$

2) See section 3 of Douglis [2].

Then, we get

Lemma 2. *If $p<\delta r$, $q<\delta' r$, then*

$$\sum_{\substack{|m|p\leqq X\\|n|q\leqq Y}}[|w_{m,n}^k|+|z_{m,n}^k|]4pq \tag{2.4}$$
$$\leqq\left[\sum_{\substack{|m|p\leqq X+\delta kr\\|n|q\leqq Y+\delta' kr}}(|w_{m,n}^0|+|z_{m,n}^0|)4pq+\frac{D}{C}\right]e^{Ckr}-\frac{D}{C},$$

where $D=4(X+\delta kr)(Y+\delta' kr)(D'+D'')$,

$$D'=\frac{p}{q}\cdot\max_{\Omega}|g_{xx}|+\max_{\Omega}|g_{xy}|+\max_{\Omega}|h_x|+2\cdot\max_{\Omega}|f_{xx}|,$$

$$D''=\frac{p}{p}\cdot\max_{\Omega}|f_{yy}|+\max_{\Omega}|f_{xy}|+\max_{\Omega}|h_y|+2\cdot\max_{\Omega}|g_{yy}|,$$

and $C=\max(\max_{\Omega}|f_{xu}|+\max_{\Omega}|f_{yu}|+\max_{\Omega}|h_u|,$

$\max_{\Omega}|g_{xu}|+\max_{\Omega}|g_{yu}|+\max_{\Omega}|h_u|)$.

This lemma asserts that if an initial grid function $u_{m,n}^0$ has locally bounded variation the solution $u_{m,n}^k$ of (2.1) with initial data $u_{m,n}^0$ also has locally bounded variation for each fixed time level.

From Lemma 2 it follows

Lemma 3. *If $p<\delta r$, $q<\delta' r$,* then, *for an even $k-j$*

$$\sum_{\substack{|m|p\leqq X\\|n|q\leqq Y}}|u_{m,n}^k-u_{m,n}^j|4pq\leqq(k-j)rL, \tag{2.5}$$

and, for an odd $k-j$,

$$\sum_{\substack{|m|p\leqq X\\|n|q\leqq Y}}|u_{m,n}^k-u_{m-1,n-1}^j|4pq\leqq(k-j)rL,\qquad j=0,1,\cdots k-1, \tag{2.6}$$

where $L=2\cdot\max(\delta,\delta')K+4(X+\delta r)(Y+\delta' r)[V^1(M)+V^2(M)+\alpha]$, *and K is the right hand side of* (2.4).

This lemma can be proved in a similar way to Lemma 4 of Conway and Smoller [1].

See also Oleinik [3; Lemma 4].

3. Proofs of the theorem. On the basis of three lemmas obtained in the last section, one can prove the theorem in the same way as in the section 3 of Conway and Smoller [1], except for Lemma 7 there.

Consider a solution $u_{m,n}^k$ of (2.1) over S_1 as a step function defined by

$$U(t,x,y)=u_{m,n}^k \tag{3.1}$$

for $t^k\leqq t<t^{k+1}$, $x_m\leqq x<x_{m+2}$, $y_n\leqq y<y_{n+2}$, $(t^k,x_m,y_n)\in S_1$.

Then, we have consequently that, if $u_0\in F$, then there exists a sequence $\{U^i(t,x,y)\}_{i=1}^{\infty}$ of solutions of (2.1) such that for each fixed t, $0\leqq t\leqq T$, $U^i(t,x,y)$ converges to some $u(t,x,y)\in F$ in the sense of L_1 over any compact set in R^2 uniformly with respect to t, where $u(t,x,y)$ have locally bounded variation in the sense of Tonelli-Cesari in G, and such that $U^i(0,x,y)$ converges to $u_0(x,y)$ in the topology of L_1 on compacta in R^2.

Therefore, in order to prove the theorem, it remains only to show that $u(t, x, y)$ thus obtained is a weak solution of (1.4), (1.5). This is an immediate consequence of the following lemma.

Lemma 4. *The function $u(t, x, y)$ satisfy the relation*

$$(1.6)\quad \iiint_G [u\varphi_t + f\varphi_x + g\varphi_y - h\varphi]dxdydt + \iint_{t=0} u_0(x, y)\varphi(0, x, y)dxdy = 0,$$

for any C^3 function $\varphi = \varphi(t, x, y)$, equal to zero outside a finite region, also for $t=T$.

We can prove this lemma by means of a device used by Oleinik in Lemma 7 of [3].

4. Concluding remarks. 1. As in Douglis [2], without loss of generality, we can make f_u^i, $i=1, \cdots, n$, nonnegative for $|u| \leqq M$ and for (t, x) in G under a suitable change of independent variables. In such a case, instead of (2.1), we may use the following difference scheme

$$(2.1)'\quad \frac{1}{r}(u_{m,n}^{k+1} - u_{m,n}^k) + \frac{1}{p}(f_{m,n}^k - f_{m-1,n}^k) + \frac{1}{q}(g_{m,n}^k - g_{m,n-1}^k) + h_{m,n}^k = 0.$$

2. In the case $n=2$, if $f^1 = f^2$ and $f_{uu}^1 \geqq 0$, and $f_{uu}^1 > \mu > 0$ for bounded u and for $\tau \geqq t \geqq 0$, where μ and τ are certain positive numbers, then one can establish that the weak solution obtained here satisfies the relation

$$\frac{u(t, x+d, y+d) - u(t, x, y)}{d} \leqq \frac{E}{t},$$

for some constant $E > 0$, and for any d.

This inequality is obtained in a similar way to the proof of lemma 2 in Oleinik [3] by setting $p=q$ in (2.1).

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