96. On Theorems of Korovkin. II

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1. P. P. Korovkin [2; Th. 3] established, among many others, the following theorem:

THEOREM 1. *Let* L_n *be a positive linear operator which maps the space* $C[a, b]$ *of all functions continuous on the closed interval* $[a, b]$ *into itself for every* $n=1, 2, \cdots$. *If*

$$(1) \qquad \lim_{n\to\infty} L_n f=f, \qquad \textit{uniformly},$$

is satisfied by $f(t)=1$, t *and* t^2, *then* (1) *is true for every* $f\in C[a, b]$.

Since several concrete operators on $C[a, b]$ are positive and linear, Korovkin's theorem plays fundamental role in his theory of approximation; for example, the Bernstein operator

$$B_n f(t)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k(1-t)^{n-k}$$

is linear and positive on $[0, 1]$ for every $n>0$.

One of the proofs of Theorem 1 due to Korovkin is based on the following theorem [2; Th. 1] on the convergence of positive linear functionals on $C[a, b]$:

THEOREM 2. *If a sequence* $\{\varphi_n\}$ *of positive linear functionals on* $C[a, b]$ *satisfies*

$$(2) \qquad \lim_{n\to\infty} \varphi_n(1)=1,$$

and

$$\lim_{n\to\infty} \varphi_n(h)=0,$$

where $h(t)=(t-c)^2$, $a\leqq c\leqq b$, *then*

$$\lim_{n\to\infty} \varphi_n(f)=f(c),$$

for all $f\in C[a, b]$.

2. A few years ago, Marie and Hisashi Choda proved in [1] an abstract version of Theorem 2. To introduce their theorem, some elementary notions on B^*-algebras are required, cf. [3].

A commutative Banach algebra A is called a *B^*-algebra* if A has an involution $x\to x^*$ which satisfies $\|xx^*\|=\|x\|^2$ for all $x\in A$. An element of A is called *positive*, symbolically $a\geqq 0$, if there is an element $b\in A$ such as $a=bb^*$. If a transformation L which maps A into a B^*-algebra B is called *positive* if $La\geqq 0$ for every $a\geqq 0$. A *character* of A is a homomorphism of A onto complex numbers. A character of A determines uniquely a maximal ideal of A.

The theorem of M. and H. Choda is as follows:

THEOREM 3. *Let A be a commutative B^*-algebra with the identity, M a principal maximal ideal generated by an element a of A, and χ the character corresponding to M. If a sequence σ_n of positive linear functionals of A satisfies* (2) *and*

$$\lim_{n\to\infty} \sigma_n(|a|^2)=0, \tag{3}$$

then σ_n converges weakly to χ:*

$$\lim_{n\to\infty} \sigma_n(x)=\chi(x), \tag{4}$$

for all $x \in A$.

Since $C[a, b]$ is a B^*-algebra with the identity, since $\chi(f)=f(c)$ determines a maximal ideal M of all continuous functions vanishing at c, and since M is generated by $a(t)=t-c$, Theorem 3 implies Theorem 2.

In the present short note, an abstract formulation of Korovkin's Theorem 1 will be given in a manner corresponding to Theorem 3.

3. An abstract version of Korovkin's Theorem 1 is the following

THEOREM 4. *Let A be a commutative B^*-algebra with the identity such that every maximal ideal is principal, and let $a_1, a_2, \cdots, a_n$ be a set of elements of A which satisfies the following property: For a given maximal ideal M of A, there exists an element g of A such that g generates M and $|g|^2=gg^*$ is expressible as a linear combination of $a_1, a_2, \cdots, a_n$,* i.e.,

$$|g|^2=\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_n a_n .$$

If a sequence L_m of linear operators satisfies

(i) L_m *maps A into itself*,

(ii) L_m *is positive,*

and

(iii) $\lim_{m\to\infty} L_m a_i=a_i$, *for* $i=0,1,2, \cdots, n$,

where $a_0=1$, then

$$\lim_{m\to\infty} L_m a=a, \tag{5}$$

for all $a \in A$.

When $A=C[a, b]$, then the requirements of the theorem are satisfied for

$$g=t-c, \quad a_1=1, \quad a_2=t, \quad \text{and} \quad a_3=t^2.$$

Hence Theorem 4 implies Theorem 1.

4. The proof of Theorem 4 is a verbal version of the second proof of Theorem 1 due to Korovkin.

Suppose the contrary that (5) fails for an element a of A. Then there exists a sequence $\{\chi_k\}$ of characters of A for which

$$|\chi_k(L_{n_k}a)-\chi_k(a)| \geqq \varepsilon>0, \tag{6}$$

where $n_1<n_2<\cdots$. Since a bounded set of the conjugate space of a Banach space is sequentially weakly* precompact, (being replaced by a subsequence if necessary) it can be assumed that χ_k converges weakly* to a certain character χ:

$$\lim_{k\to\infty}\chi_k(x)=\chi(x), \tag{7}$$

for all $x\in A$. It will be also assumed that g generates the maximal ideal M corresponding to χ. By (iii), L_n converges strongly at $h=|g|^2$ since h is a linear combination of $a_1, a_2, \cdots, a_n$.

Put

$$\sigma_k(x)=\chi_k(L_{n_k}x), \qquad k=1, 2, \cdots \tag{8}$$

Since L_n is positive and linear, σ_k is also positive and linear, and (2) is automatically satisfied by $\{\sigma_k\}$ since L_n1 converges to 1. Furthermore, (3) is also satisfied by $\{\sigma_k\}$, since χ_k converges to χ uniformly on $\{L_nh\}$ by (iii) and (7) so that

$$\lim_{k\to\infty}\sigma_k(h)=\lim_{k\to\infty}\chi_k(L_{n_k}h)=\chi(h)=0.$$

Hence σ_k converges weakly* to χ; especially,

$$\lim_{k\to\infty}\sigma_k(a)=(\chi a). \tag{9}$$

Now, by (7), (8), and (9),

$$|\chi_k(L_{n_k}a)-\chi_k(a)|\leqq|\chi_k(L_{n_k}a)-\chi(a)|+|\chi(a)-\chi_k(a)|<\varepsilon,$$

for sufficiently large k, which contradicts to (6). This proves the theorem.

References

[1] H. Choda and M. Echigo: On theorems of Korovkin. Proc. Japan Acad., **39**, 107-108 (1963).

[2] P. P. Korovkin: Linear Operators and Approximation Theory. Hindustan, Delhi (1960).

[3] C. E. Rickart: General Theory of Banach Algebras. Van Nostrand, Princeton (1960).