21. On a Certain Family of Meromorphic Functions

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§1. Introduction. Let M be all the non-constant functions each of which is one-valued and meromorphic in $|z|<+\infty$, and we put $\varphi(x, y)=|f(z)|=|f(x+iy)|$ where $z=x+iy$ (x, y real) and $f\in M$. Now we consider the following two conditions:

(C. 1.) There exists a properly chosen vicinity V where $\varphi\neq 0$, $\varphi_y \neq 0$, and $\dfrac{\varphi_x}{\varphi_y}=A(x)B(y)$, $A(x)$, $B(y)$ being a function of x and a function of y respectively.

(C. 2.) There exists a properly chosen vicinity V where $\varphi\neq 0$, $\varphi_x \neq 0$, and $\dfrac{\varphi_y}{\varphi_x}=A(x)B(y)$, $A(x)$, $B(y)$ being a function of x and a function of y respectively.

We consider the set of all the functions each of which satisfies either the condition (C. 1.) or the condition (C. 2.) and denote this set by S. For example, z^n $(n=\pm 1, \pm 2, \pm 3, \cdots)$, $\exp z$, $\exp(iz)$, $\exp(\exp z)$, $\exp(i\exp z)$, $\sin z$, $\cos z$, $\tan z$, $\cot z$, $\sec z$, $\operatorname{cosec} z$, $\sinh z$, $\cosh z$, $\tanh z$, $\coth z$, $\operatorname{sech} z$, $\operatorname{cosech} z$, $sn(z, k)$, $cn(z, k)$, $dn(z, k)$, $sc(z, k)=\dfrac{sn(z, k)}{cn(z, k)}$, $cs(z, k)=\dfrac{cn(z, k)}{sn(z, k)}$, $ns(z, k)=\dfrac{1}{sn(z, k)}$, $nc(z, k)=\dfrac{1}{cn(z, k)}$, $nd(z, k)=\dfrac{1}{dn(z, k)}$ belong to S. (k is a real constant.)

§2. A necessary and sufficient condition that $f(z)(\in M)$ belongs to S.

Lemma (see [3]). Let D be a simply connected domain. If $u^{(1)}(x, y)$, $u^{(2)}(x, y)$, $u^{(1)}(x, y)u^{(2)}(x, y)$ are harmonic in D and the conjugate harmonic functions of $u^{(1)}(x, y)$, $u^{(2)}(x, y)$ in D are $v^{(1)}(x, y)$, $v^{(2)}(x, y)$ respectively and if we put $f_1(z)=u^{(1)}(x, y)+iv^{(1)}(x, y)$ ($f_1(z) \not\equiv$const. in D), $f_2(z)=u^{(2)}(x, y)+iv^{(2)}(x, y)$, then in D $f_2(z)=i\alpha f_1(z)+\beta$ where α is a real constant and β is a complex constant.

Proof. Since this is easy, we omit it.

Theorem 1. A function $f(z)(\in M)$ belongs to S if and only if there exists a function $g(z)$ which satisfies the five following conditions:

(i) $g(z)$ is one-valued and meromorphic in $|z|<+\infty$.
(ii) $g(z)\not\equiv 0$. ($g(z)$ may be a non-zero complex constant.)
(iii) The residue of $g(z)$ at every point is an integer.

(iv) $g(z)$ satisfies the differential equation $g'^2(z)=ag^4(z)+bg^2(z)+c$ where a, b, c are real constants.

(v) $f(z)=C\exp\left(\int g(z)dz\right)$ where C is a non-zero complex constant.

Proof. Necessity. We may assume that $f(z)$ satisfies the condition (C. 1.). Putting $f(z)=u+iv$ where u, v are real, we have in V $\dfrac{\varphi_x}{\varphi_y}=\dfrac{uu_x+vv_x}{uu_y+vv_y}=\dfrac{uu_x+vv_x}{-uv_x+vu_x}$. Putting $g(z)=\dfrac{f'(z)}{f(z)}$, we have in V $g(z)=\dfrac{uu_x+vv_x}{u^2+v^2}+i\dfrac{uv_x-vu_x}{u^2+v^2}$. Putting $p=\dfrac{uu_x+vv_x}{u^2+v^2}$, $q=\dfrac{uv_x-vu_x}{u^2+v^2}$, we have in V

$$\frac{\varphi_x}{\varphi_y}=-\frac{p}{q}. \tag{1}$$

Since $\dfrac{\varphi_x}{\varphi_y}=A(x)B(y)$ in V, we have in V

$$\frac{\varphi_x}{\varphi_y}\frac{\partial^2}{\partial y\partial x}\left(\frac{\varphi_x}{\varphi_y}\right)=\frac{\partial}{\partial x}\left(\frac{\varphi_x}{\varphi_y}\right)\frac{\partial}{\partial y}\left(\frac{\varphi_x}{\varphi_y}\right). \tag{2}$$

By (1), (2) we have in V

$$pq(pp_{xx}+qq_{xx})=p_xq_x(p^2+q^2). \tag{3}$$

Since $g(z)\not\equiv 0$ in V, there exists a subvicinity V_1 ($\subseteqq V$) where $g(z)\neq 0$. Hence, by (3) we have in V_1

$$pq\frac{pp_{xx}+qq_{xx}}{p^2+q^2}=p_xq_x. \tag{4}$$

We may assume that $g(z)$ is not a complex constant. Since $pq=Im\left(\dfrac{1}{2}g^2\right)=Re\left(\dfrac{1}{2i}g^2\right)$ $\left(\dfrac{1}{2i}g^2\not\equiv\text{const. in } V_1\right)$, $\dfrac{pp_{xx}+qq_{xx}}{p^2+q^2}=Re\left(\dfrac{g''}{g}\right)$ and $p_xq_x=Im\left(\dfrac{1}{2}g'^2\right)$, pq, $\dfrac{pp_{xx}+qq_{xx}}{p^2+q^2}$ and p_xq_x are harmonic in V_1. By the above Lemma we have in V_1

$$\frac{g''}{g}=i\alpha\left(\frac{1}{2i}g^2\right)+\beta=\frac{1}{2}\alpha g^2+\beta, \tag{5}$$

where α is a real constant and β is a complex constant.

By (5) we have in V_1 $pq\dfrac{pp_{xx}+qq_{xx}}{p^2+q^2}=Im\left(\dfrac{1}{2}g^2\right)Re\left(\dfrac{g''}{g}\right)=Im\left(\dfrac{1}{2}g^2\right)Re\left(\dfrac{1}{2}\alpha g^2+\beta\right)=Im\left\{\dfrac{1}{8}\alpha g^4+\dfrac{1}{2}Re(\beta)g^2\right\}$. Hence we have in V_1

$$pq\frac{pp_{xx}+qq_{xx}}{p^2+q^2}=Im\left\{\frac{1}{8}\alpha g^4+\frac{1}{2}Re(\beta)g^2\right\}. \tag{6}$$

By (4), (6) we have in V_1 $\dfrac{1}{2}g'^2=\dfrac{1}{8}\alpha g^4+\dfrac{1}{2}Re(\beta)g^2+\dfrac{1}{2}c$ where c is a real constant. Putting $a=\frac{1}{4}\alpha$, $b=Re(\beta)$, we have in V_1 $g'^2=ag^4+bg^2+c$ where a, b, c are real constants.

On the other hand (i), (ii), (iii), (v) are clearly satisfied.

Sufficiency. By (ii) there exists a properly chosen vicinity V where $g(z)\neq 0$. We have in V $pq\frac{pp_{xx}+qq_{xx}}{p^2+q^2}-p_xq_x=Im\left(\frac{1}{2}g^2\right)Re\left(\frac{g''}{g}\right)-Im\left(\frac{1}{2}g'^2\right)$. We may assume that $g(z)$ is not a complex constant. Since we have $g'^2=ag^4+bg^2+c$, by differentiation ($g\neq$const.) we have $g''=2ag^3+bg$. Putting $g=p+iq$ where p, q are real, we have in V

$$
\begin{aligned}
pq\frac{pp_{xx}+qq_{xx}}{p^2+q^2}-p_xq_x&=Im\left(\frac{1}{2}g^2\right)Re\left(\frac{g''}{g}\right)-Im\left(\frac{1}{2}g'^2\right)\\
&=Im\left(\frac{1}{2}g^2\right)Re\left(\frac{2ag^3+bg}{g}\right)-Im\left(\frac{1}{2}ag^4+\frac{1}{2}bg^2+\frac{1}{2}c\right)\\
&=Im\left(\frac{1}{2}g^2\right)Re(2ag^2)+Im\left(\frac{1}{2}bg^2\right)-Im\left(\frac{1}{2}ag^4+\frac{1}{2}bg^2\right)\\
&=Im\left(\frac{1}{2}ag^4\right)-Im\left(\frac{1}{2}ag^4\right)=0.
\end{aligned}
$$

Hence we have $f(z)\in S$.

§ 3. Determination of the family S.

Theorem 2. The functions which belong to S are the following and only these:

$$
\text{(i)}\quad f(z)=C\left(\frac{\mu+\lambda sn^2\left(\frac{1}{2}\left(\frac{\mu}{n}z+A\right),\frac{\lambda}{\mu}\right)}{\mu-\lambda sn^2\left(\frac{1}{2}\left(\frac{\mu}{n}z+A\right),\frac{\lambda}{\mu}\right)}\right)^n \quad (n=\pm 1, \pm 2, \pm 3,\cdots),
$$

where C ($\neq 0$), A are arbitrary complex constants and λ^2, μ^2 are the two roots of the quadratic equation $at^2+bt+c=0$ with real coefficients a ($\neq 0$), b, c ($\neq 0$).

(ii) $f(z)=C\exp(A\sin\alpha z+B\cos\alpha z)$,

or $f(z)=C\exp(A\sinh\alpha z+B\cosh\alpha z)$,

or $f(z)\equiv C\exp(A_1z^2+A_2z+A_3)$,

where C ($\neq 0$), A, B ($|A|+|B|>0$), A_2, A_3 are arbitrary complex constants and α ($\neq 0$) is an arbitrary real constant and A_1 ($|A_1|+|A_2|>0$) is a real or purely imaginary constant.

(iii) $f(z)=C\tan^n(\alpha z+A)$ $(n=\pm 1, \pm 2, \pm 3,\cdots)$,

or $f(z)=C\tanh^n(\alpha z+A)$ $(n=\pm 1, \pm 2, \pm 3,\cdots)$,

or $f(z)=(Cz+A)^n$ $(n=\pm 1, \pm 2, \pm 3,\cdots)$,

where C ($\neq 0$), A are arbitrary complex constants and α ($\neq 0$) is a real constant.

Proof.

$$
(1)\qquad g'^2=ag^4+bg^2+c
$$

where a, b, c are real constants.

(i) $a\neq 0$, $c\neq 0$. Then we have $g'^2=a(\lambda^2-g^2)(\mu^2-g^2)$, where $\lambda^2\,(\neq 0)$, $\mu^2\,(\neq 0)$ are the two roots of the quadratic equation $at^2+bt+c=0$. We may assume that $g(z)\neq$const.

Putting $h(z)=\frac{1}{\lambda}g\left(\frac{z}{\sqrt{a}\,\mu}\right)$, we have $h'^2=(1-h^2)(1-k^2h^2)$ where $k^2=\frac{\lambda^2}{\mu^2}$. Hence we have $h(z)=\pm sn(z+A, k)(h(z)\neq\text{const.})$ where A is a complex constant. Hence we have

(2) $$g(z)=\pm\lambda sn(\sqrt{a}\,\mu z+A, k).$$

Since $sn(z, k)$ $(k\neq 0)$ has a (simple) pole with residue $\frac{1}{k}$ at $z=iK'$, $g(z)$ has a (simple) pole with residue $\pm\frac{1}{\sqrt{a}}$ at $z=\frac{iK'-A}{\sqrt{a}\,\mu}$. Hence, by the property (iii) of $g(z)$ we have $\pm\frac{1}{\sqrt{a}}=n$ where $n\,(\neq 0)$ is an integer. Hence we have $a=\frac{1}{n^2}$. Hence, by (2) we have $g(z)=\lambda sn\left(\frac{\mu}{n}z+A,\ k\right)$ where A is an arbitrary complex constant and $n(\neq 0)$ is an arbitrary integer.

Since $\int sn(z,k)dz=\frac{1}{k}\log\frac{1+k\,sn^2\left(\frac{1}{2}z,\ k\right)}{1-k\,sn^2\left(\frac{1}{2}z,\ k\right)}$ $(k\neq 0)$, we have

$$f(z)=C\left(\frac{\mu+\lambda sn^2\left(\frac{1}{2}\left(\frac{\mu}{n}z+A\right), \frac{\lambda}{\mu}\right)}{\mu-\lambda sn^2\left(\frac{1}{2}\left(\frac{\mu}{n}z+A\right), \frac{\lambda}{\mu}\right)}\right)^n.$$

(ii) $a=0$. Then we have the solutions (ii).

(iii) $c=0$. Then we have the solutions (iii).

Corollary. The entire functions which belong to S are the following and only these:

(i) $$f(z)=C(\cosh(\alpha z+A))^n \quad (n=1, 2, 3, \cdots),$$
where $C\,(\neq 0)$, A are arbitrary complex constants and $\alpha\,(\neq 0)$ is an arbitrary real constant.

(ii) $$f(z)=C\exp(A\sin\alpha z+B\cos\alpha z),$$
or $$f(z)=C\exp(A\sinh\alpha z+B\cosh\alpha z),$$
or $$f(z)=C\exp(A_1z^2+A_2z+A_3),$$
where $C\,(\neq 0)$, A, B $(|A|+|B|>0)$, A_2, A_3 are arbitrary complex constants and $\alpha\,(\neq 0)$ is a real constant and A_1 $(|A_1|+|A_2|>0)$ is a real or purely imaginary constant.

(iii) $$f(z)=(Cz+A)^n \quad (n=1, 2, 3, \cdots),$$
where $C(\neq 0)$, A are arbitrary complex constants.

Proof. Since this is easy, we omit it.

§ 4. Some applications to the functional equations.

Theorem 3. Let us assume that $f(z)$ is an entire function of z and satisfies the following functional equation:

$$(1)\qquad |f(x+iy)|^2=|f(x)|^2+|f(iy)|^2,$$

where x, y are real, then such functions as $f(z)$ are Cz, $C\sin\alpha z$, $C\sinh\alpha z$ where C is an arbitrary complex constant and α is an arbitrary real constant. The solutions are only these.

Proof. By (1) we have $f(z)\in S$. Since $f(z)$ is an entire function of z, by Corollary of Theorem 2 the assertion is proved.

Corollary. Let us assume that $f(z)$ is an entire function of z which is real on the real axis and satisfies the following functional equation:

$$(1)\qquad f(x+y)f(x-y)=f^2(x)-f^2(y),$$

then such functions as $f(z)$ are Cz, $C\sin\alpha z$, $C\sinh\alpha z$ where C, α are arbitrary real constants. The solutions are only these.

Proof. If s, t are real, by (1) we have

$$\begin{aligned}|f(s+it)|^2&=f(s+it)\overline{f(s+it)}=f(s+it)f(s-it)\\&=f^2(s)-f^2(it)=|f(s)|^2+|f(it)|^2.\end{aligned}$$

Theorem 4. Let us assume that a function $f(z)\in M$ satisfies the following functional equation (see [1]) (k is a real constant):

$$(1)\qquad |f(x+iy)|^2=\frac{|f(x)|^2+|f(iy)|^2}{1+k^2|f(x)|^2|f(iy)|^2}(x,\ y\ \text{real}).$$

(i) If $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-(1+k^2)$, $f^{(4)}(0)=0$ and $f^{(5)}(0)=1+14k^2+k^4$, then $f(z)=sn(z,k)$.

(ii) If $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=1+k^2$, $f^{(4)}(0)=0$ and $f^{(5)}(0)=1+14k^2+k^4$, then $f(z)=sc(z,k')\left(=\dfrac{sn(z,k')}{cn(z,k')}\right)(k^2+k'^2=1)$.

Proof. The proof is similar to that of Theorem 3.

Corollary. Let us assume that a function $f(z)\in M$ satisfies the following functional equation (k is a real constant) and $f(z)$ is real on the real axis:

$$(1)\qquad f(x+y)f(x-y)=\frac{f^2(x)-f^2(y)}{1-k^2f^2(x)f^2(y)}\ (x,\ y\ \text{complex}).$$

(i) If $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-(1+k^2)$, $f^{(4)}(0)=0$ and $f^{(5)}(0)=1+14k^2+k^4$, then $f(z)=sn(z,k)$.

(ii) If $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=1+k^2$, $f^{(4)}(0)=0$ and $f^{(5)}(0)=1+14k^2+k^4$, then $f(z)=sc(z,k')$.

Proof. The proof is similar to that of Corollary of Theorem 3.

In the similar manner, under the suitable conditions, we can systematically discuss the solutions of many functional equations from the standpoint of the family S.

Let us assume that $u=\varphi(x,y)$ ($\not\equiv$const.) is one-valued and harmonic in the whole xy-plane. Now we consider the two following conditions:

(C. 1.) There exists a properly chosen vicinity V where $\varphi_y \neq 0$ and $\frac{\varphi_x}{\varphi_y} = A(x)B(y)$, $A(x)$, $B(y)$ being a function of x and a function of y respectively.

(C. 2.) There exists a properly chosen vicinity V where $\varphi_x \neq 0$ and $\frac{\varphi_y}{\varphi_x} = A(x)B(y)$, $A(x)$, $B(y)$ being a function of x and a function of y respectively.

We consider the set of all the functions each of which is one-valued and harmonic in the whole xy-plane and satisfies either the condition (C. 1.) or the condition (C. 2.), and denote this set by T.

Theorem 5. The functions which belong to T are the following and only these:

$$\varphi(x, y) = Re(A \sin \alpha z + B \cos \alpha z + C),$$

or

$$\varphi(x, y) = Re(A \sinh \alpha z + B \cosh \alpha z + C),$$

or

$$\varphi(x, y) = Re(A_1 z^2 + A_2 z + A_3),$$

where $C\ (\neq 0)$, A, B $(|A| + |B| > 0)$, A_2, A_3 are arbitrary complex constants and $\alpha\ (\neq 0)$ is a real constant and A_1 $(|A_1| + |A_2| > 0)$ is a real or purely imaginary constant.

Proof. Considering the entire function $f(z)$ of $z = x + iy$ such that $Re\{f(z)\} = \varphi(x, y)$, we have $\exp(f(z)) \in S$. By Corollary of Theorem 2 the assertion is proved.

By Theorem 5, under the hypotheses that $f(z)$ is an entire function of z and is real on the real axis, we can systematically discuss the solutions of many functional equations from the standpoint of the family T. For example, considering the functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$, we have $Re\{f(s+it)\} \in T$ (s, t real).

(This article is dedicated to Prof. Kunugi on the occasion of his 60th birthday.)

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