141. On the Cauchy Problem for a Class of Multicomponent Diffusion Systems

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Introduction. This note discusses the Cauchy problem for a class of multicomponent diffusion systems of the form

$$(1)\qquad \begin{aligned} \Lambda u &= f(x, t, u, v), \\ \partial v/\partial t &= g(x, t, u, v), \end{aligned}$$

where $x=(x_1,\cdots,x_n)$ and Λ is a linear parabolic differential operator:

$$\Lambda u \equiv \partial u/\partial t - \Big[\sum_{i,j=1}^{n} a_{ij}(x, t)\partial^2 u/\partial x_i \partial x_j + \sum_{i=1}^{n} b_i(x, t)\partial u/\partial x_i + c(x, t)u\Big].$$

Let E^n denote the n-dimensional Euclidean x-space and H the strip $H=E^n\times(0, T]$, $T>0$, in the $(n+1)$-dimensional (x, t)-space.

By the Cauchy problem in question we mean the problem of finding function pairs $\{u(x, t),\ v(x, t)\}$ which are continuous in $\overline{H}$, satisfy the system (1) in H and take on the given initial values:

$$(2)\qquad u(x, 0)=\varphi(x) \quad v(x, 0)=\psi(x), \quad x\in E^n.$$

Our main concern in this note is with the comparison (§1) and the existence (§2) of solutions of the problem (1)–(2), being suggested by an elegant work of A. McNabb [1] on the first boundary value problem for the system (1) in cylindrical domains.[1)]

Preliminary hypotheses. The following assumptions concerning the system (1) will be made throughout the note:

1) *The coefficients a_{ij}, b_i and c are defined and continuous in $\overline{H}$;*

2) *At each point $(x, t)\in\overline{H}$ and for all real n-tuples $\xi=(\xi_1\cdots,\xi_n)$,*

$$(3)\qquad \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqq a_0\sum_{i=1}^{n}\xi_i^2, \quad (a_0\text{: a positive constant});$$

3) *The functions f and g are defined in the domain $\mathcal{D}=\{(x,t)\in\overline{H}, -\infty<u<\infty,\ -\infty<v<\infty\}$ and are subject to the conditions:*

i) *f is a non-decreasing function of v, while g is a non-decreasing function of u;*

ii) *Both f and g are uniformly Lipschitz continuous relative to u and v:*

$$(4)\qquad |h(x, t, u, v)-h(x, t, \overline{u}, \overline{v})| \leqq M(|u-\overline{u}|+|v-\overline{v}|),$$

for (x, t, u, v), $(x, t, \overline{u}, \overline{v})\in\mathcal{D}$ with $h=f$ or g.

§1. Comparison theorems.

To begin with, the following spaces of function pairs $\{u(x, t\},\ v(x, t)\}$ defined in $\overline{H}$ are introduced.

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1) We also refer to the works of V. N. Maslennikova [2], [3].

$\mathfrak{F}$: the set of all function pairs $\{u, v\}$, the u-components of which have continuous second x-derivatives and continuous first t-derivatives in $\overline{H}$, while the v-components of which are continuous with their first t-derivatives in $\overline{H}$.

$\mathfrak{A}$: the set of those function pair $\{u, v\} \in \mathfrak{F}$, the u's of which satisfy the inequality $|u(x, t)| \leqq m$ in $\overline{H}$, m being a constant.

$\mathfrak{B}$: the set of those function pairs $\{u, v\} \in \mathfrak{F}$, the u's of which satisfy the inequality $|u(x, t)| \leqq m(1+r^p)$ in $\overline{H}$, where $r=\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ and m and p are positive constants.

$\mathfrak{C}$: the set of those function pairs $\{u, v\} \in \mathfrak{F}$, the u's of which satisfy the inequality $|u(x, t)| \leqq \exp(\beta(1+r^2))$, β being a constant>0.

Thoerem 1. *Suppose that the following inequalities are valid:*[2]

(5) $\quad |a_{ij}(x, t)| \leqq M(1+r^2), \quad |b_i(x, t)| \leqq M(1+r^2)^{1/2}, \quad c(x, t) \leqq M.$

And suppose further that the function pairs $\{u_1, v_1\}, \{u_2, v_2\} \in \mathfrak{A}$ *satisfy the system of inequalities*:

(6)
$$\Lambda u_1 - f(x, t, u_1, v_1) \leqq \Lambda u_2 - f(x, t, u_2, v_2),$$
$$\partial v_1/\partial t - g(x, t, u_1, v_1) \leqq \partial v_2/\partial t - g(x, t, u_2, v_2), \text{ in } H,$$

(7)
$$u_1(x, 0) \leqq u_2(x, 0), \quad v_1(x, 0) \leqq v_2(x, 0), \quad x \in E^n.$$

Then, we conclude that

(8)
$$u_1(x, t) \leqq u_2(x, t), \quad v_1(x, t) \leqq v_2(x, t) \text{ in } \overline{H}.$$

Proof. Define an auxiliary function pair $\{U, V\}$ by

$$U(x, t) = u_2(x, t) + (2m/r_0^2)(r^2+Kt)e^{\alpha t},$$
$$V(x, t) = v_2(x, t) + (2m/r_0^2)(r^2+Kt)e^{\alpha t},$$

where K, r_0 and α are positive constants.

An easy computation shows, in view of (6), that

$$\Lambda U - f(x, t, U, V) \geqq \Lambda u_1 - f(x, t, u_1, v_1),$$
$$\partial V/\partial t - g(x, t, U, V) \geqq \partial v_1/\partial t - g(x, t, u_1, v_1),$$

providing $\alpha \geqq (4n+3)M$ and $K \geqq (4n+3)M$. Now consider $\{U, V\}$ in the cylinder Q_{r_0} $(r \leqq r_0, 0 \leqq t \leqq T)$. Since obviously $U \geqq u_1$ on the normal boundary of Q_{r_0} and $V \geqq v_1$ on the lower basis of Q_{r_0}, we may apply McNabb's theorem [1, Theorem 2], concluding therefore that

(9) $\quad U(x, t) \geqq u_1(x, t), \quad V(x, t) \geqq v_1(x, t)$ in Q_{r_0} for all r_0.

Any fixed point $(x, t) \in \overline{H}$ enters the cylinder Q_{r_0} for all sufficiently large r_0 and at that point the inequalities (9) are valid. Letting r_0 tend to infinity in (9) we obtain

$$u_2(x, t) \geqq u_1(x, t), \quad v_2(x, t) \geqq v_1(x, t),$$

thus completing the proof.

Remark. We note that in Theorem 1 the assumption that $\{u_i, v_i\} \in \mathfrak{A}$ $(i=1, 2)$ can be replaced by the requirement that they belong

2) Of course, the assumptions 1), 2), and 3) given in the introduction are tacitly made in this and the subsequent theorems.

to the class $\mathfrak{B}$. To prove this, it is only necessary to introduce another pair of functions

$$U(x, t)=u_2(x, t)+(4m/r_0^{2q-p})(r^2+Kt)^q e^{\alpha t},$$
$$V(x, t)=v_2(x, t)+(4m/r_0^{2q-p})(r^2+Kt)^q e^{\alpha t}, \quad 2q>p,$$

and to repeat the arguments employed there.

Theorem 2. *In this theorem assume that*

$$(10) \qquad |a_{ij}(x, t)| \leqq M, \quad |b_i(x, t)| \leqq M, \quad |c(x, t)| \leqq M.$$

Let the function pairs $\{u_1, v_1\}, \{u_2, v_2\} \in \mathfrak{C}$ *satisfy the system of inequalities* (6)–(7). *Then, the assertion of Theorem 1 also holds.*

Proof. The auxiliary functions to be employed here are

$$U(x, t)=u_2(x, t)+2 \exp(2\beta(1+r^2)e^{\alpha t}-\beta(1+r_0^2)),$$
$$V(x, t)=v_2(x, t)+2 \exp(2\beta(1+r^2)e^{\alpha t}-\beta(1+r_0^2)).$$

A simple computation establishes that

$$\Lambda U-f(x, t, U, V) \geqq \Lambda u_1-f(x, t, u_1, v_1),$$
$$\partial V/\partial t-g(x, t, U, V) \geqq \partial v_1/\partial t-g(x, t, u_1, v_1)$$

for $0 \leqq t \leqq 1/\alpha$ if we take $\alpha=2M(4\beta ne+n+2/\beta)$.

If follows therefore, exactly as in the preceding proof, that

$$u_2(x, t) \geqq u_1(x, t), \quad v_2(x, t) \geqq v_1(x, t)$$

in the strip $0 \leqq t \leqq 1/\alpha$. Continuing the same arguments for the strip $1/\alpha \leqq t \leqq 2/\alpha$, and then, for the strip $2/\alpha \leqq t \leqq 3/\alpha$, etc., we finally conclude that the desired inequalities (8) hold throughout $\overline{H}$.

Remark. From each of the above theorems there results a uniqueness theorem that insures the unicity of solutions of the Cauchy problem (1)–(2) in the corresponding space of function pairs. We shall not, however, give rather obvious statements of those uniqueness theorems.

§2. **Existence theorem. Theorem 3.** *Let the following assumptions be made*:

I) *The coefficients* a_{ij}, b_i, *and* c *are bounded and uniformly Hölder continuous (exponent* λ*) in* $\overline{H}$ *relative to the parabolic metric* $d(P, \overline{P})=(|x-\overline{x}|^2+|t-\overline{t}|)^{1/2}$, *where* $P=(x, t)$, $\overline{P}=(\overline{x}, \overline{t}) \in \overline{H}$. *Moreover,* a_{ij} *are uniformly Hölder continuous (exponent* λ*) in* $\overline{H}$ *relative to the usual Euclidean metric* $\rho(P, \overline{P})=(|x-\overline{x}|^2+|t-\overline{t}|^2)^{1/2}$.[3)]

II) *The functions* f *and* g *are bounded and uniformly Hölder continuous (exponent* λ*) in* $\overline{H}$ *for each fixed value of* $\{u, v\}$.

III) *The function* $\varphi(x)$ *is bounded and Hölder continuous (exponent* λ*) in* E^n *together with its second derivatives, while the function* $\psi(x)$ *is bounded and Hölder continuous (exponent* λ*) in* E^n.

Suppose further that there exist two pairs $\{\overline{\Phi}, \overline{\Psi}\}$, $\{\underline{\Phi}, \underline{\Psi}\}$ *of func-*

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3) From now on, the Hölder continuity of functions of (x, t) is to be understood in the sense of the parabolic metric.

tions, bounded and uniformly Hölder continuous (exponent λ) in $\bar{H}$ and satisfying the system of inequalities[4)]

$$\begin{aligned} &\Lambda\underline{\Phi}-f(x, t, \underline{\Phi}, \underline{\Psi})\leqq 0\leqq \Lambda\overline{\Phi}-f(x, t, \overline{\Phi}, \overline{\Psi}),\\ &\partial\underline{\Psi}/\partial t-g(x, t, \underline{\Phi}, \underline{\Psi})\leqq 0\leqq \partial\overline{\Psi}/\partial t-g(x, t, \overline{\Phi}, \overline{\Psi}), \end{aligned} \tag{11}$$

$$\underline{\Phi}(x, 0)\leqq\varphi(x)\leqq\overline{\Phi}(x, 0),\quad \underline{\Psi}(x, 0)\leqq\psi(x)\leqq\overline{\Psi}(x, 0). \tag{12}$$

Under these assumptions there exists a unique bounded solution $\{u, v\}$ *of the Cauchy problem* (1)–(2).

Proof. The existence-proof will be carried out with the aid of the method of iterations as in McNabb's paper. We proceed as follows.

a) *The iteration scheme.* Consider the sequence of function pairs $\{u_n, v_n\}$ $(n=1, 2\cdots)$ defined by the equations

$$\begin{aligned} &\Lambda u_{n+1}+Mu_{n+1}=f(x, t, u_n, v_n)+Mu_n,\\ &\partial v_{n+1}/\partial t+Mv_{n+1}=g(x, t, u_n, v_n)+Mv_n; \end{aligned} \tag{13}$$

$$u_{n+1}(x, 0)=\varphi(x),\quad v_{n+1}(x, 0)=\psi(x);\quad u_1=\overline{\Phi},\quad v_1=\overline{\Psi}. \tag{14}$$

If u_n and v_n are known to be uniformly Hölder continuous with exponent λ in $\bar{H}$, then so are the right hand sides of (13). From the existence theorem on the Cauchy problem for linear parabolic equations (see, e.g., the reference [4]) and from the famous theorem of A. Friedman [5] it readily follows that the function u_{n+1} can be determined uniquely and is uniformly Hölder continuous with the same exponent in $\bar{H}$. The existence and the Hölder continuity of v_{n+1} are implied by the explicit formula

$$e^{Mt}v_{n+1}(x, t)=\psi(x)+\int_0^t e^{M\tau}(g(x, \tau, u_n, v_n)+Mv_n)\, d\tau. \tag{15}$$

b) *The monotony of the sequences* $\{u_n\}$ *and* $\{v_n\}$, $(n=1, 2,\cdots)$.

An induction with the aid of the comparison theorems of §1 establishes that

$$\begin{aligned} &\underline{\Phi}(x, t)\leqq u_{n+1}(x, t)\leqq u_n(x, t)\leqq\overline{\Phi}(x, t),\\ &\underline{\Psi}(x, t)\leqq v_{n+1}(x, t)\leqq v_n(x, t)\leqq\overline{\Psi}(x, t),\quad n=1, 2,\cdots . \end{aligned}$$

We set $u(x, t)=\lim_{n\to\infty} u_n(x, t)$ and $v(x, t)=\lim_{n\to\infty} v_n(x, t)$.

c) *The Hölder continuity of* u *and* v. Noting the uniform boundedness of u_n and v_n and using A. Friedman's theorem, the uniform Hölder continuity of u can easily be verified. On the other hand, that v is also uniformly Hölder continuous in $\bar{H}$ can be proven by means of the integration formula (15) and a modification of the arguments of A. McNabb.

d) Once the uniform Hölder continuity of u and v has been established, we are now able to solve the system

4) If $|f(x, t, u, v)|\leqq A+B|u|+C|v|$, $|g(x, t, u, v)|\leqq A'+B'|u|+C'|v|$, then $\overline{\Phi}=\overline{\Psi}=-\underline{\Phi}=-\underline{\Psi}=\alpha e^{\beta t}$ satisfy (11) and (12) providing α and β are taken large enough.

$$\Lambda u^{\#}+Mu^{\#}=f(x, t, u, v)+Mu,$$
$$\partial v^{\#}/\partial t+Mv^{\#}=g(x, t, u, v)+Mv,$$
$$u^{\#}(x, 0)=\varphi(x), \quad v^{\#}(x, 0)=\psi(x).$$

e) The identical coincidence: $u^{\#}(x, t)\equiv u(x, t)$, $v^{\#}(x, t)\equiv v(x, t)$ in $\bar{H}$ is easily concluded by considering the equations satisfied by $u^{\#}-u_n$ and by $v^{\#}-v_n$. It thus follows that the pair $\{u, v\}$ is assuredly the solution of our Cauchy problem (1)–(2). This completes the proof.

References

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