78. On the Uniqueness of Solutions of the Cauchy Problem for Hypoelliptic Partial Differential Operators

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(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1963)

1. Introduction. L. Härmander in the note [2] proved the following: For a differential operator L with constant coefficients the uniqueness of solutions for the Cauchy problem does not hold in the class C^∞, if the initial plane is characteristic for L.

The object of this note is to prove the uniqueness of solutions for the Cauchy problem, whose initial plane may be characteristic, under the restriction of the class of possible solutions which belong to $\{u; \exp(-(1+|x|^2)^{1/2})\cdot u \in C^m_{[0,T]}(\mathfrak{H}_{2m})\}$ defined in the section 2. Let $R^{1+\nu}$ be the $(1+\nu)$-dimensional Euclidean space with coordinates $(t, x) =(t, x_1,\cdots, x_\nu)$, and $(m, \mathfrak{m})=(m, m_1,\cdots, m_\nu)$ $(m_j \leqq 2m;\ j=1,\cdots,\nu)$ be an appropriate real vector whose elements are positive integers. We shall consider differential operators L of the form

$$L=\sum_{i/m+|\alpha:\mathfrak{m}|=1} a_{i,\alpha}(t,x)\frac{\partial^{i+|\alpha|}}{\partial t^i \partial x^\alpha} \quad (a_{m,0}(t,x)=1) \tag{1.1}$$

$$(\alpha=(\alpha_1,\cdots,\alpha_\nu),\ x^\alpha=x_1^{\alpha_1}\cdots x_\nu^{\alpha_\nu},$$
$$|\alpha|=\alpha_1+\cdots\alpha_\nu,\ |\alpha:\mathfrak{m}|=\alpha_1/m_1+\cdots+\alpha_\nu/m_\nu)$$

where $a_{i,\alpha}(t,x)$ belong to $\mathscr{B}_{(t,x)}$ in $[0, T]\times R^\nu$. Here we remark this class is an extension of the result of S. Mizohata [6] and contains the operators of the form

$$L=(-1)^{[s/2]}\frac{\partial^s}{\partial t^s}+(-1)^m\sum_{|\alpha|=2m}A_\alpha(t,x)\frac{\partial^{2m}}{\partial x^\alpha}{}^{1)} \tag{1.2}$$

where $s\geqq m$ and $\sum_{|\alpha|=2m}A_\alpha(t,x)\xi^\alpha\geqq\delta>0$ for $|\xi|=1$; see [5].

2. Definitions and lemmas. We set the associated polynomials $L(t,x,\lambda,\xi)$ of (1.1) for real vectors $(\lambda,\xi)=(\lambda,\xi_1,\cdots,\xi_\nu)$

$$L(t,x,\lambda,\xi)=\sum_{i/m+|\alpha:\mathfrak{m}|=1}a_{i,\alpha}(t,x)\lambda^i\xi^\alpha \quad (a_{m,0}(t,x)=1). \tag{2.1}$$

Let us define $r=r(\xi)$ as a positive root of the equation $\sum_{j=1}^{\nu}\xi_j^2 r^{-2/m_j}=1$ $(\xi\neq 0)$, and set $K(\xi)=\{\sum_{j=1}^{\nu}\xi_j^{2m_j}\}^{1/4m}$. Then, we have

$$\nu^{1/4m}K(\xi)\leqq r^{1/2m}\leqq\nu^{1/2}K(\xi),\ |\partial^{|\alpha|}/\partial\xi^\alpha r(\xi)^{1/2m}|\leqq C_\alpha{}^{2)}K(\xi)^{1-2m|\alpha:\mathfrak{m}|}. \tag{2.2}$$

The proof is given in [4], but in this case we must replace m by $2m$.

We denote by $\mathfrak{H}_p$ a function space

$$\mathfrak{H}_p=\left\{u\in L^2(R^\nu);\ ||u||_p^2=\int(1+K(\xi))^{2p}|\hat{u}(\xi)|^2d\xi<\infty\right\},$$

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1) For these operators, $a_i(t)$ in (3.1) of this note are constants.

2) In what follows constants C are always positive.

where $\hat{u}(\xi)=\frac{1}{\sqrt{2\pi}^{\nu}}\int e^{-\sqrt{-1}x\cdot\xi}u(x)\,dx$ (Fourier transform of u), $u(t)=u(t,x)\in C^{k}_{[0,T]}(\mathfrak{H}_p)$ means that $u(t)$ belongs to $\mathfrak{H}_p$ and k-times continuously differentiable in the topology of $\mathfrak{H}_p$.

If we write for real vectors $\eta=(\eta_1,\cdots,\eta_\nu)\neq 0$, $L(t,x,\lambda,\sqrt{-1}\eta|\eta|^{-1})=\prod_{i=1}^{m}(\lambda+\lambda_{0,i}(t,x,\eta))$ and define a matrix R by

$$R=\begin{pmatrix} r^{1/m_1} & & 0 \\ & \ddots & \\ 0 & & r^{1/m_\nu}\end{pmatrix}.$$

Then, we can write

$$L(t,x,\lambda,\sqrt{-1}\,\xi)=\prod_{i=1}^{m}(\lambda+\lambda_{0,i}(t,x,\xi R^{-1})r^{1/m});\quad \text{see [4].} \tag{2.3}$$

Definition 1. We call H a singular integral operator of class $C^{2m}_{\mathfrak{m}}$ with the symbol $\sigma(H)(x,\xi)=\sum_{r=1}^{\infty}a_r(x)\hat{h}_r(\xi)$ $(a_r(x)\in\mathscr{B}_{(x)},\ \hat{h}_r(\xi)\in C^{\infty}_{(\xi\neq 0)};\ r=1,2,\cdots)$ if we have for every l, α and β

$$|\partial^{|\alpha|+|\beta|}/\partial x^\alpha\partial\xi^\beta\, a_r(x)\hat{h}_r(\xi)|\leqq C_{l,\alpha,\beta}r^{-l}K(\xi)^{-2m|\beta:\mathfrak{m}|}\quad (r=1,2,\cdots).$$

Then, for $u\in L^2=\mathfrak{H}_0$ Hu is defined by

$$Hu=\frac{1}{\sqrt{2\pi}^{\nu}}\int e^{\sqrt{-1}x\cdot\xi}\sigma(H)(x,\xi)\hat{u}(\xi)\,d\xi.$$

Definition 2. An operator Λ is defined by $\widehat{\Lambda u}(\xi)=\hat{\Lambda}(\xi)\hat{u}(\xi)$ for $u\in\mathfrak{H}_1$, where $\hat{\Lambda}(\xi)\in C^{\infty}_{(\xi\neq 0)}$ and for every α satisfies the condition

$$|\partial^{|\alpha|}/\partial\xi^\alpha\hat{\Lambda}(\xi)|\leqq C_\alpha K(\xi)^{1-2m|\alpha:\mathfrak{m}|}.$$

If $\lambda_0(x,\eta)$ $(\in C^{\infty}_{(x,\eta)}\cap\mathscr{B}_{(x)}$ for $\eta\neq 0)$ is homogeneous of order zero in η, then by [1] we can expand it as $\lambda_0(x,\eta)=\sum_{r=1}^{\infty}a_r(x)\hat{h}_{0,r}(\eta)$. Hence, if we set $\hat{h}_r(\xi)=\hat{h}_{0,r}(\xi R^{-1})(r=1,2,\cdots)$, we can write $\lambda_0(x,\xi R^{-1})=\sum_{r=1}^{\infty}a_r(x)\hat{h}_r(\xi)$ which becomes the symbol of an operator of class $C^{2m}_{\mathfrak{m}}$; see [4]. Furthermore, if we set $\hat{\Lambda}(\xi)=r(\xi)^{1/2m}$, then by (2.2) we can define an operator Λ.

For the operators defined in the above definitions the following lemma is easily verified by essentially same methods with [7] and [8] if we remark $2m|\alpha:\mathfrak{m}|\geqq|\alpha|$ by assumption $m_j\leqq 2m$ $(j=1,\cdots,\nu)$.

Lemma 1. i) *Let P be an operator of class $C^{2m}_{\mathfrak{m}}$ with the real valued symbol. Then $P\Lambda-\Lambda P^{*}$* [3] *is bounded in $\mathfrak{H}_0$.*

ii) *For H, H_1 and H_2 $(\in C^{2m}_{\mathfrak{m}})$, we have for any positive integer p and q the following representations:*

(2.4) $H\Lambda^p-\Lambda^p H=H_{p,q}\Lambda^{p-1}+H'_{p,q}$, $(H_1H_2-H_1\circ H_2)$[4] $\Lambda=H_q+H'_q$,
where $H_{p,q}$, $H_q\in C^{2m}_{\mathfrak{m}}$ and $\Lambda^i H'_{p,q}\Lambda^j$, $\Lambda^i H'_q\Lambda^j$ $(0\leqq i+j\leqq q)$ are bounded in $\mathfrak{H}_p$.

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3) P^* is the adjoint operator of P.

4) $H_1\circ H_2$ means an operator of class $C^{2m}_{\mathfrak{m}}$ with $\sigma(H_1\circ H_2)=\sigma(H_1)\sigma(H_2)$.

iii) *For the operator H $(\in C^{2m}_{\mathfrak{M}})$ such as $|\sigma(H)|\geqq\delta>0$, we have*

$$||HA^p u||^2\geqq\frac{1}{2}\delta^2||HA^p u||^2-C||u||^2. \tag{2.5}$$

Lemma 2. *Let $H^0=H^0(t)$ be a bounded operator defined in $\mathfrak{H}_2$ and strongly continuous together with its derivative in $[0, T]$, and for $u, v\in\mathfrak{H}_2$ satisfy the conditions*:

$$\begin{aligned} &\text{i)} \quad ||H^0u||^2\geqq\delta||\Lambda^2u||^2-C||u||^2 \quad (\delta>0),\\ &\text{ii)} \quad |H^{0\prime}u, u)^{5)}|\leqq\varepsilon||\Lambda^2u||^2+C\varepsilon^{-1}||u||^2 \quad (0<\varepsilon\leqq1),\\ &\text{iii)} \quad (H^0u, v)=(u, H^0v) \quad (\textit{self-adjoint}). \end{aligned} \tag{2.6}$$

Then, for the operator $J^0=d/dt+(1+\sqrt{-1}\,a(t))H^0(t)$ $(a(t)\in C^1_{[0,T]}$; real valued) and $u=u(t)\in C^1_{[0,T]}(\mathfrak{H}_2)$ such as $u(0)=u(h)=0$ $(0<h\leqq T)$, we have with $\varphi=1+h^{-1}t$

$$\int_0^h\varphi^{-2n}||J^0u||^2\,dt\geqq C_0\left\{h^{-1}\int_0^h\varphi^{-2n}||\Lambda u||^2\,dt+h^{-2}\int_0^h\varphi^{-2n}||u||^2\,dt\right\} \tag{2.7}$$

for a constant C_0, sufficiently small h and every $n(\geqq1)$.

Proof. We set $u=\varphi^n v$, then $v\in C^1_{[0,T]}(\mathfrak{H}_2)$ and $v(0)=v(h)=0$.

$$\int_0^h\varphi^{-2n}||J^0u||^2\,dt=\int_0^h||(v'+\sqrt{-1}\,aH^0v)+(H^0v+nh^{-1}\varphi^{-1}v)||^2\,dt$$

$$\geqq2Re\int_0^h(v'+\sqrt{-1}\,aH^0v, H^0v+nh^{-1}\varphi^{-1}v)\,dt+\int_0^h||H^0v+nh^{-1}\varphi^{-1}v||^2dt$$

and using ii) and iii) of (2.6) and $v(0)=v(h)=0$ we can continue

$$\geqq\int_0^h\frac{d}{dt}(H^0v, v)\,dt-\int_0^h(H^{0\prime}v, v)\,dt+\left\{\int_0^h nh^{-1}\varphi^{-1}||v||^2\,dt\right.$$

$$\left.+\int_0^h(||H^0v||^2-2nh^{-1}\varphi^{-1}||H^0v||\,||v||+n^2h^{-2}\varphi^{-2}||v||^2\,dt\right\}$$

$$\geqq-\int_0^h(\varepsilon||\Lambda^2v||^2+C\varepsilon^{-1}||v||^2)\,dt+\frac{1}{4}\int_0^h\left(\frac{1}{n}||H^0v||^2+nh^{-2}\varphi^{-2}||v||^2\right)dt.$$

Now, if we set $\varepsilon=\delta/8n$ and apply i) of (2.6) to H^0v, then remarking $\varphi^{-2}\geqq1/4$ $(0\leqq t\leqq h)$ we have for sufficiently small h

$$\int_0^h\varphi^{-2n}||J^0u||^2\,dt\geqq\frac{\delta}{8n}\int_0^h||\Lambda^2v||^2\,dt+\frac{1}{20}nh^{-2}\int_0^h||v||^2\,dt.$$

As $h^{-1}||\Lambda v||^2\leqq n^{-1}||\Lambda^2v||^2+nh^{-2}||v||^2$ and $v=\varphi^{-n}u$, we get (2.7).

Q.E.D.

Lemma 3. *Let $H_i(t)$ $(i=1,\cdots,k)$ be of class $C^{2m}_{\mathfrak{M}}$ with t as a parameter and $(k-1)$-times continuously differentiable in $[0, T]$, and*

$$|\sigma(H_i-H_j)|\geqq\delta>0 \quad (i\neq j).$$

Then, setting $J_i=d/dt+H_i(t)\Lambda^2$ we have for constants C and C' and $u\in C^{k-1}_{[0,T]}(\mathfrak{H}_{2(k-1)})$

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5) (,) means the inner product in $L^2(R^\nu)$.

$$\begin{aligned}(2.8)\qquad &\sum_{i_1,\cdots,i_{k-1}} \| J_{i_1}\bullet\cdots\bullet J_{i_{k-1}} u\|^2 \\ &\geqq C\sum_{i+j/2=k-1}\left\|\frac{d^i}{dt^i}\Lambda^j u\right\|^2 - C'\sum_{i+j/2\leqq k-3/2}\left\|\frac{d^i}{dt^i}\Lambda^j u\right\|^2,\end{aligned}$$

where $J_{i_1},\cdots,J_{i_{k-1}}$ run all permutations from $J_1,\cdots,J_k$.

Proof is essentially the same with that of Lemma 4 in [3].

Lemma 4. *Let $H_i(t)$ $(i=1,\cdots,k)$ be the operators of Lemma 3. Suppose for each $H_i(t)$ there exists an operator $H_i^0(t)$ which satisfies the assumptions of Lemma 2 and*

$$(2.9)\qquad \|(H_i(t)\Lambda^2-(1+\sqrt{-1}\,a_i(t))H_i^0(t))u\|^2 \leqq C(\|\Lambda u\|^2+\|u\|^2)\quad (u\in\mathfrak{H}_2)$$

for some real valued function $a_i(t)\in C^1_{[0,T]}$. Then, for the operator $A=J_1\bullet\cdots\bullet J_k$ $(J_i=d/dt+H_i\Lambda^2;\ i=1,\cdots,k)$ and $u\in C^k_{[0,T]}(\mathfrak{H}_{2k})$ such as $d^{j-1}/dt^{j-1}u(0)=d^{j-1}/dt^{j-1}u(h)=0$ $(j=1,\cdots,k)$, we get

$$(2.10)\qquad \int_0^h \varphi^{-2n}\|Au\|^2\,dt \geqq C_0\sum_{i+j/2=\tau\leqq k-1/2} h^{-2(k-\tau)}\int_0^h \varphi^{-2n}\left\|\frac{d^i}{dt^i}\Lambda^j u\right\|^2 dt$$

for a constant C_0, sufficiently small h and every $n\geqq 1$.

Proof. If we write $J_i=(d/dt+(1+\sqrt{-1}\,a_i)H_i^0)+(H_i\Lambda^2-(1+\sqrt{-1}\,a_i)H_i^0)$, then, by the assumption of H_i^0 we can easily verify the inequality (2.7) holds even for J_i. This means (2.10) holds for $k=1$. Next estimating $(J_1\bullet\cdots\bullet J_k - J_{i_1}\bullet\cdots\bullet J_{i_k})u$ by (2.4) we get

$$\begin{aligned}&\|J_1\bullet\cdots\bullet J_k u\|^2 \\ &\geqq C_1\sum_{i_1,\cdots,i_k}\|J_{i_1}\bullet\cdots\bullet J_{i_k}u\|^2 - C_2\sum_{i+j/2\leqq k-1/2}\left\|\frac{d^i}{dt^i}\Lambda^j u\right\|^2.\end{aligned}$$

If we apply (2.7) to each $J_{i_1}(J_{i_2}\bullet\cdots\bullet J_{i_k}u)$, we have

$$\begin{aligned}\int_0^h \varphi^{-2n}\|J_1\bullet\cdots\bullet J_k u\|^2\,dt &\geqq C_3\sum_{i_1,\cdots,i_k} h^{-1}\int_0^h \varphi^{-2n}\|\Lambda J_{i_2}\bullet\cdots\bullet J_{i_k}u\|^2\,dt \\ &+C_3h^{-2}\int_0^h\varphi^{-2n}\|J_2\bullet\cdots\bullet J_k u\|^2\,dt - C_4\sum_{i+j/2\leqq k-1/2}\int_0^h \varphi^{-2n}\left\|\frac{d^i}{dt^i}\Lambda^j u\right\|^2 dt.\end{aligned}$$

Estimating $(\Lambda J_{i_2}\bullet\cdots\bullet J_{i_k} - J_{i_2}\bullet\cdots\bullet J_{i_k}\Lambda)u$ by (2.4) we apply Lemma 3 to the first term, and for the second term we use (2.10) replacing k by $(k-1)$ as the assumption of the induction. Then, we get (2.10) for sufficiently small h. Q.E.D.

3. Main theorems. Theorem 1. *Let L be an operator defined in* (1.1). *Suppose the associated polynomial $L(t,x,\lambda,\xi)$ can be written as*

$$(3.1)\qquad L(t,x,\lambda,\sqrt{-1}\,\eta|\eta|^{-1})=\prod_{i=1}^m(\lambda+(1+\sqrt{-1}\,a_i(t))\lambda_{0,i}(t,x,\eta))\quad(\eta\neq 0)$$

where $a_i(t)$ $(\in C^m_{[0,T]})$ and $\lambda_{0,i}(t,x,\eta)$ $(\in C^\infty_{(t,x,\eta)}\cap\mathscr{B}_{(t,x)}$ for $\eta\neq 0)$ are all real valued functions, and $|\lambda_{0,i}(t,x,\eta)|\geqq\delta>0$ and

$|(1+\sqrt{-1}\,a_i(t))\lambda_{0,i}(t,x,\eta)-(1+\sqrt{-1}\,a_j(t))\lambda_{0,j}(t,x,\eta)|\geqq\delta>0$ $(i\neq j)$ for $\eta\neq 0$. Then, we have for a constant C

$$(3.2)\qquad \int_0^h\varphi^{-2n}\|Lu\|^2\,dt\geqq C\sum_{i/m+|\alpha:\mathfrak{m}|=\tau\leqq 1/2m} h^{-2m(1-\tau)}\int_0^h\varphi^{-2n}\left\|\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}u\right\|^2 dt$$

for sufficiently small h and every $n\geqq 1$, where $\varphi=1+h^{-1}t$ and $u\in C^m_{[0,T]}(\mathfrak{H}_{2m})$ such as $\partial^{j-1}/\partial t^{j-1}u(0)=\partial^{j-1}/\partial t^{j-1}u(h)=0$ $(j=1,\cdots,m)$.

Proof. By the discussion of the previous section we can write

$$L(t,x,\lambda,\sqrt{-1}\,\xi)=\prod_{i=1}^{m}(\lambda+(1+\sqrt{-1}\,a_i(t))\lambda_{0,i}(t,x,\xi R^{-1})r^{1/m}).$$

Now, we consider operators H_i and P_i of class $C^{2m}_{\mathfrak{m}}$ with $\sigma(H_i)=(1+\sqrt{-1}\,a_i(t))(\lambda_{0,i}(t,x,\xi R^{-1})$ and $\sqrt{\lambda_{0,i}(t,x,\xi R^{-1})}$ respectively, and consider an operator Λ defined by $\hat{\Lambda}(\xi)=r(\xi)^{1/2m}$. Then, if we set $H_i^0=\Lambda P_i{}^*P_i\Lambda$, using Lemma 1 fully we can verify each pair H_i and H_i^0 satisfies the conditions of Lemma 4. Hence, setting $J_i=d/dt+H_i\Lambda^2$ $(i=1,\cdots,m)$ we get (2.10) as $k=m$.

Consequently, estimating commutators we have

$$\int_0^h\varphi^{-2n}\|Lu\|^2\,dt\geqq C_1\sum_{i+j/2=\tau\leqq m-1/2}h^{-2(m-\tau)}\int_0^h\varphi^{-2n}\left\|\frac{d^i}{dt^i}\Lambda^j u\right\|^2dt. \tag{3.3}$$

As $h^a r^{a/m}\leqq h^j r^{j/m}+1$ $(0\leqq a\leqq j)$ we have $h^a\|\Lambda^a u\|^2\leqq h^j\|\Lambda^j u\|^2+\|u\|^2$, and

$$\left\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u\right\|^2=\int|\xi^\alpha|^2|\hat{u}(\xi)|^2\,d\xi\leqq\int(K(\xi)^{2m|\alpha:\mathfrak{m}|})^2|\hat{u}(\xi)|^2\,d\xi$$

$$\leqq C_2\int(r^{2m|\alpha:\mathfrak{m}|/2m})^2|\hat{u}(\xi)|^2\,d\xi=C_2\|\Lambda^{2m|\alpha:\mathfrak{m}|}u\|^2.$$

Hence, by (3.3) we have for sufficiently small h

$$\int_0^h\varphi^{-2n}\|Lu\|^2\,dt\geqq C_3\sum_{i+m|\alpha:\mathfrak{m}|=\tau\leqq m-1/2}h^{-2(m-\tau)}\int_0^h\varphi^{-2n}\left\|\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}u\right\|^2dt.$$

Replacing τ by $m\tau$ we get (3.2).

Theorem 2. *Let L be an operator which satisfies the conditions of Theorem 1.*

Suppose $u=u(t,x)$ belongs to $L^2(R^\nu)$ with t as a parameter and $\exp(-r_0(x))u(t,x)\in C^m_{[0,T]}(\mathfrak{H}_{2m})$ $\left(r_0(x)=\left\{1+\sum_{j=1}^{\nu}x_j^2\right\}^{1/2}\right)$.

Then, if u satisfies a differential inequality

$$|Lu|^2\leqq C\sum_{i/m+|\alpha:\mathfrak{m}|\leqq 1-1/2m}\left|\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}u\right|^2 \tag{3.4}$$

and initial conditions

$$\left.\frac{\partial^{j-1}}{\partial t^{j-1}}u(t,x)\right|_{t=0}=0;\qquad j=1,\cdots,m, \tag{3.5}$$

u vanishes identically in $[0,T]$.

Proof. It is clear that $|\partial^{|\alpha|}/\partial x^\alpha r_0(x)|(|\alpha:\mathfrak{m}|\leqq 1)$ are uniformly bounded, hence if we set $v=\exp(-r_0(x))u$, v also satisfies the condition (3.4) and (3.5).

Take $\psi(t)\in C^\infty_{[0,T]}$ such that $\psi(t)=1$ for $0\leqq t\leqq h/2$ and zero for $3/4h\leqq t\leqq h$, and set $w=\psi v$. Then, we can apply the inequality (3.2) to w and get

$$\int_0^h\varphi^{-2n}\|Lw\|^2\,dt\geqq C_1\sum_{i/m+|\alpha:\mathfrak{m}|=\tau\leqq 1-1/2m}h^{-2m(1-\tau)}\int_0^h\varphi^{-2n}\left\|\frac{\partial^{i+|\alpha|}}{\partial t^i\partial x^\alpha}u\right\|^2dt.$$

As $Lw=Lv$ in $[0, h/2]$, we use (3.4) for v and get

$$C \sum_{i/m+|\alpha:\mathfrak{m}|\leqq 1-1/2m} \int_0^h \varphi^{-2n} \left\| \frac{\partial^{i+|\alpha|}}{\partial t^i \partial x^\alpha} u \right\|^2 dt + \int_{h/2}^h \varphi^{-2n} \| Lw \|^2 dt$$

$$\geqq C_1 \sum_{i/m+|\alpha:\mathfrak{m}|=\tau\leqq 1-1/2m} h^{-2m(1-\tau)} \int_0^{h/2} \varphi^{-2n} \left\| \frac{\partial^{i+|\alpha|}}{\partial t^i \partial x^\alpha} u \right\|^2 dt.$$

Hence, for sufficiently small fixed h we get

$$\int_{h/2}^h \varphi^{-2n} \| Lw \|^2 dt \geqq \frac{C_1}{2} h^{-2m} \int_0^{h/3} \varphi^{-2n} \| v \|^2 dt.$$

As $\varphi^{-2n} \leqq (2/3)^{2n}$ for $h/2 \leqq t \leqq h$ and $\varphi^{-2n} \geqq (3/4)^{2n}$ for $0 \leqq t \leqq h/3$, we get $\left(\frac{8}{9}\right)^{2n} \int_{h/2}^h \| Lw \|^2 dt \geqq \frac{C_1}{2} h^{-2m} \int_0^{h/3} \| v \|^2 dt$. Letting $n \to \infty$ we get $v=u=0$ on $[0, h/3]$. Repeating this process we get $u(t) \equiv 0$ in $[0, T]$. Q.E.D.

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