51. *Remarks on Some Properties of Solutions of Some Boundary Value Problems for Quasi-linear Parabolic and Elliptic Equations of the Second Order*

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Introduction. In this note we shall try to generalize some of the results established by Oleinik [6] and Výborný [7] for linear elliptic and parabolic differential equations of the second order. Namely, we shall consider second order quasi-linear parabolic and elliptic equations and discuss first the behavior of their solutions at the boundary of the domain where they attain positive maximum or negative minimum. Next we shall formulate the uniqueness theorems for some boundary value problems with oblique derivatives. In our discussion extensive use is made of the maximum principles proved by the author [8], [9] for quasi-linear elliptic and parabolic equations. Since the treatment is similar for both parabolic and elliptic cases, we shall limit ourselves in our exposition to the detailed consideration of parabolic equations, while for elliptic equations only the corresponding theorems will be stated.

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§1. Quasi-linear parabolic equations. In this section we are concerned with quasi-linear parabolic equations of the form

$$(1) \qquad \sum_{i,j=1}^{n} a_{ij}(x, t, u, \text{grad } u)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f(x, t, u, \text{grad } u),$$

$$x=(x_1, \cdots, x_n), \text{ grad } u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n).$$

We denote by D a bounded domain in the $(n+1)$-dimensional (x, t)-space bounded by two hyperplanes $t=0$ and $t=T>0$, and by a lateral surface S lying between these hyperplanes. The union of the surface S and the lower basis $B=\overline{D}\frown\{t=0\}$ is referred to as the normal boundary of D and is denoted by ∂D. We assume that the functions $a_{ij}(x, t, u, p)$ and $f(x, t, u, p)$ are defined in the domain $\mathfrak{D}$: $\{(x, t)\in D, |u|<\infty, ||p||<\infty\}$ and are bounded in compact subset of $\mathfrak{D}$. We impose the following assumption on the lateral surface S of D: for each point $P(x, t)\in S$ there exists an $(n+1)$-dimensional sphere K_P including P on its boundary such that all the points of K_P lying in the strip $0<t\leqq T$ belong to $\overline{D}-\partial D$. Finally we assign to each point of S a direction l which makes an acute angle with the inwardly

directed normal n to S at that point.

Theorem 1. *Let $u(x,t)$ be continuous in $\overline{D}$ and satisfy the equation in $\overline{D}-\partial D$. Let $P_1(x_1,t_1)\in S$ be a point where the solution $u(x,t)$ attains its positive maximum in $\overline{D}$. Then we have either $u\equiv$const. in some neighborhood of P_1 for $t<t_1$ or*

$$(2) \qquad \limsup_{P\to P_1}\frac{u(P)-u(P_1)}{r(P,P_1)}<0$$

where $r(P,P_1)$ is the distance between P and P_1 and P approaches P_1 along the direction l mentioned above, provided that the following assumptions are satisfied:

I) *There exists a positive lower semi-continuous function $h(x,t,u,p)$ such that*

$$\sum_{i,j=1}^{n}a_{ij}(x,t,u,p)\xi_i\xi_j\geqq h(x,t,u,p)||\xi||^2$$

for every $(x,t,u,p)\in\mathfrak{D}$ and for every real vector ξ;

II) *$f(x,t,u,0)\geqq 0$ for $u\geqq 0$;*

III) *$f(x,t,u,p)$ satisfies locally the Lipschitz condition with respect to u and p.*

Proof. Let $t_1<T$. The case $t_1=T$ can be treated similarly. We find a sphere $K_{P_1}\subset\overline{D}$ with radius R which touches the lateral surface S at P_1. It follows from the maximum principle ([9], Theorem 3) that $u(P)<u(P_1)$ in the interior of K_{P_1} provided $u(x,t)$ is not constant in some neighborhood of P_1 for $t<t_1$. We may assume that the center of K_{P_1} coincides with the origin of the coordinate system. Draw a sphere K with center P_1 and radius less than R and set $\omega=K\cap K_{P_1}$. Define the function $v_k(x,t)$ by

$$v_k(x,t)=u(x,t)-u(x_1,t_1)+\varepsilon[\exp(-k(x^2+t^2))-\exp(-kR^2)],$$

k and ε being positive constants. It is clear that $v_k(x,t)$ is non-positive on the boundary of ω for sufficiently small ε: more precisely, v_k equals 0 at P_1 and is negative elsewhere. Our assertion is that for such ε and for suitably chosen k $v_k(x,t)$ is negative in ω. This assertion may be verified by means of an argument analogous to that employed by the author [8], [9]. In fact, assume for contradiction that $m_k=\max_{\overline{D}} v_k$ is positive for every k and let P_k be an interior point of ω such that $v_k(P_k)=m_k$. Applying the parabolic differential operator

$$\mathcal{P}=\sum_{i,j=1}^{n}a_{ij}(x,t,u(x,t),\ \text{grad}\ u(x,t))\frac{\partial^2}{\partial x_i\partial x_j}-\frac{\partial}{\partial t}$$

to functions v_k and noting that they are maximal at P_k we have first $\mathcal{P}v_k(P_k)\leqq 0$. By means of the device used by the author we have on the other hand $\mathcal{P}v_k(P_k)>0$ for sufficiently large k. The contradiction thus obtained proves our assertion from which the desired inequality immediately follows:

$$\limsup_{P\to P_1}\frac{u(P)-u(P_1)}{r(P, P_1)} \leqq -\varepsilon\frac{\partial v_k(P_1)}{\partial l}$$
$$=-2\varepsilon k \exp\left(-k(||x_1||^2+t_1^2)\right)\sqrt{||x_1||^2+t_1^2}\cos(n, l)<0.$$

Theorem 2. *We consider the equation* (1) *under the following assumptions*:

I) $\sum_{i,j=1}^{n} a_{ij}(x, t, u, p)\xi_i\xi_j \geqq h(x, t, u, p)||\xi||^2$;

II) $\operatorname{sign} u\cdot f(x, t, u, 0)\geqq 0$ *and* $f(x, t, 0, 0)\equiv 0$;

III) $f(x, t, u, p)$ *satisfies locally the Lipschitz condition with respect to* u *and* p.

We assume that a solution $u(x,t)$ *of* (1) *continuous in* $\overline{D}$ *satisfies the boundary conditions*

$$a\frac{\partial u}{\partial l}+bu=0 \quad on\ S, \tag{3}$$

$$u(x, 0)=0, \tag{4}$$

where $a\geqq 0, b\leqq 0$ *and* $|a|+|b|>0$ *on* S.

Under these assumptions we conclude that $u(x, t)$ *vanishes identically in* $\overline{D}$.

Remark. If the direction l lies on the hyperplanes $t=$const. then the assertion of Theorem 2 is valid under the same assumptions as in Theorem 2 except that the condition II) is replaced by a less restrictive condition

II′) $f(x, t, 0, 0)\equiv 0$.

Proof. Assume that $u\not\equiv 0$ in D. Without loss of generality we may assume that $m=\max_{\overline{D}} u(x, t)>0$. Let P_1 be a point where $u(P_1)=m$. From the strong maximum principle it follows that P_1 necessarily belongs to S. The boundary conditions (3), (4) imply that $\frac{\partial u(P_1)}{\partial l}\geqq 0$. Hence $u(x, t)$ must be a constant in a neighborhood of P_1 for $t<t_1$ by virtue of Theorem 1. Joining P_1 and a point P_2 on the lower basis by a curve in D along which the t coordinates vary monotonically and applying the strong maximum principle we conclude finally that $u(P_2)=m$ which contradicts the condition (4).

Theorem 3. *The quasi-linear parabolic equation* (1) *possesses at most one solution which is continuous in* $\overline{D}$, *bounded in* $\overline{D}$ *with its derivatives appearing in* (1) *and satisfies the boundary conditions*

$$a\frac{\partial u}{\partial l}+bu=\varphi \quad on\ S, \tag{5}$$

$$u(x, 0)=\psi(x), \tag{6}$$

provided that the following restrictions are satisfied.:

I) $\sum_{i,j=1}^{n} a_{ij}(x, t, u, p)\xi_i\xi_j \geqq h(x, t, u, p)||\xi||^2$;

II) $a_{ij}(x, t, u, p)$ *and* $f(x, t, u, p)$ *satisfy locally the Lipschitz condition with respect to* u *and* p;

III) $a \geqq 0, b \leqq 0$ *and* $|a|+|b|>0$.

Proof. Let $u(x, t)$ and $u_0(x, t)$ be two solutions of one and the same problem (1), (5), and (6). Then the difference $v=u-u_0$ evidently satisfies a quasi-linear parabolic equation

$$(1') \qquad \sum_{i,j=1}^{n} A_{ij}(x, t, v, \operatorname{grad} v)\frac{\partial^2 v}{\partial x_i \partial x_j} - \frac{\partial v}{\partial t} = F(x, t, v, \operatorname{grad} v),$$

where

$$A_{ij}(x, t, v, \operatorname{grad} v) = a_{ij}(x, t, v+u_0, \operatorname{grad} v + \operatorname{grad} u_0)$$

and

$$F(x, t, v, \operatorname{grad} v) = f(x, t, v+u_0, \operatorname{grad} v + \operatorname{grad} u_0) - f(x, t, u_0, \operatorname{grad} u_0)$$
$$- \sum_{i,j=1}^{n} (a_{ij}(x, t, v+u_0, \operatorname{grad} v + \operatorname{grad} u_0) - a_{ij}(x, t, u_0, \operatorname{grad} u_0))\frac{\partial^2 u_0}{\partial x_i \partial x_j}.$$

Applying Theorem 2 to the equation (1′) we can conclude that $v \equiv 0$ and hence that $u \equiv u_0$ in D.

§ 2. **Quasi-linear elliptic equations.** In this section we consider second order quasi-linear elliptic equations of the form

$$(7) \qquad \sum_{i,j=1}^{n} a_{ij}(x, u, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u),$$

where the functions $a_{ij}(x, u, p)$ and $f(x, u, p)$ are defined in some domain $\mathcal{G}$: $\{x \in G, |u| < \infty, ||p|| < \infty\}$, ($G$: a bounded domain in the Euclidean n-space) and are bounded in any compact subset of $\mathcal{G}$. We assume that the boundary ∂G of G has the following property: for each point $P \in \partial G$ there exists a sphere K_P contained in $\bar{G}$ whose boundary has only one point P in common with ∂G. To every point $P \in \partial G$ we assign a direction l which makes an acute angle with the inwardly directed normal n at that point.

Theorem 4. *Let the following assumptions be fulfilled:*

I) *There exists a positive lower semi-continuous function* $h(x, u, p)$ *such that*

$$\sum_{i,j=1}^{n} a_{ij}(x, u, p)\xi_i\xi_j \geqq h(x, u, p)||\xi||^2$$

for every $(x, u, p) \in \mathcal{G}$ *and every real* n*-tuple* ξ;

II) $f(x, u, 0) \geqq 0$ *for* $u \geqq 0$;

III) $f(x, u, p)$ *satisfies locally the Lipschitz condition with respect to* u *and* p.

Let further $u(x)$ *be a solution of the equation* (7) *in* G *which is continuous in* $\bar{G}$. *Then if* $u(x)$ *is not constant in* G *and assumes its non-negative maximum at some point* P_1 *on the boundary* ∂G, *we have*

$$\limsup_{P \to P_1} \frac{u(P)-u(P_1)}{r(P, P_1)} < 0 \tag{8}$$

where P approaches P_1 along the assigned direction l.

Theorem 5. *Consider the equation (7) under the following restrictions:*

I) $\sum_{i,j=1}^{n} a_{ij}(x, u, p)\xi_i\xi_j \geqq h(x, u, p)\|\xi\|^2$;

II) $f(x, 0, 0) \equiv 0$ *and* sign $u \cdot f(x, u, 0) > 0$ *for* $u \neq 0$;

III) $f(x, u, p)$ *satisfies locally the Lipschitz condition with respect to u and p.*

Let further a solution $u(x)$ continuous in $\overline{G}$ of (7) satisfy the boundary condition with an oblique derivative

$$a\frac{\partial u}{\partial l} + bu = 0 \quad on\ \partial G, \tag{9}$$

where $a \geqq 0, b \leqq 0$ and $|a| + |b| > 0$. Under these assumptions we conclude that $u(x)$ vanishes indentically in $\overline{G}$.

Theorem 6. *In this theorem we deal with the quasi-linear elliptic equation*

$$\sum_{i,j=1}^{n} a_{ij}(x, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u) \tag{10}$$

with the boundary condition of the form

$$a\frac{\partial u}{\partial l} + bu = \varphi \quad on\ \partial G. \tag{11}$$

The boundary value problem (10), (11) possesses at most one solution which is continuous in $\overline{G}$ and bounded with its derivatives appearing in (10), provided that following assumptions are satisfied:

I) *There exists a positive lower semi-continuous function $h(x, p)$ such that*

$$a_{ij}(x, p)\xi_i\xi_j \geqq h(x, p)\|\xi\|^2$$

for every (x, p) under consideration and every real vector ξ;

II) $f(x, u, p)$ *is strictly increasing with respect to u;*

III) $f(x, u, p)$ *satisfies locally the Lipschitz condition with respect to u and p;*

IV) $a_{ij}(x, p)$ *satisfy locally the Lipschitz condition with respect to p;*

V) $a \geqq 0$, $b \leqq 0$ *and* $|a| + |b| > 0$.

Theorem 7. *We now consider the equation*

$$\sum_{i,j=1}^{n} a_{ij}(x, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, \operatorname{grad} u) \tag{12}$$

with the boundary condition

$$\frac{\partial u}{\partial l} = \varphi \quad on\ \partial G \tag{13}$$

under the assumptions

I) $\sum_{i,j=1}^{n} a_{ij}(x, p)\xi_i\xi_j \geqq h(x, p)||\xi||^2$;

II) *$a_{ij}(x, p)$ and $f(x, p)$ satisfy locally the Lipschitz condition with respect to p.*

We then conclude that any two solutions of the problem (12), (13) *differ only by a constant.*

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