50. On the Maximum Principle for Quasi-linear Parabolic Equations of the Second Order

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Introduction. In this note we shall discuss the maximum-minimum property of solutions of general quasi-linear parabolic equations of the second order. For linear parabolic equations such property, known as the maximum principle, has been exhaustively exploited and has been playing an essential part in the study of both linear and non-linear parabolic equations. As is well known, the strongest results in this connection have been given by Nirenberg [4]. It seems, however, that the maximum-minimum property for quasi-linear parabolic equations has hitherto been investigated unsatisfactorily and that the deeper investigation might enable us to establish results of more or less use.

The main purpose of this note is to give an extension of the so-called "strong maximum principle" established by Nirenberg [4] to the case of quasi-linear parabolic equations. Section 2 is devoted to this extension. We note here that this is an analogue of the maximum principle proved by the author [5]. In section 1 we formulate without proofs a very simple maximum principle and some of its consequences. In both sections from the maximum principles immediately follow the uniqueness theorems for the first boundary value problem and the Harnack type convergence theorems.

Let D denote a bounded domain in the $(n+1)$-dimensional (x, t)-space, bounded by two hyperplanes $t=0$ and $t=T>0$, and by a surface S lying between these hyperplanes. $\overline{D}$ denotes the closure of D, B the lower basis of $D: B=\overline{D}\cap\{t=0\}$, and ∂D the normal boundary of D consisting of S and B.

Quasi-linear parabolic equations we are concerned with are of the type

$$(1)\qquad \sum_{i,j=1}^{n} a_{ij}(x, t, u, \operatorname{grad} u)\frac{\partial^2 u}{\partial x_i \partial x_j}-\frac{\partial u}{\partial t}=f(x, t, u, \operatorname{grad} u)$$

$$(x=(x_1,\cdots, x_n),\ \operatorname{grad} u=(\partial u/\partial x_1,\cdots, \partial u/\partial x_n)).$$

The functions $a_{ij}(x, t, u, p)$ and $f(x, t, u, p)$ are defined in the domain $\mathfrak{D}$: $\{(x, t)\in D, |u|<\infty, \|p\|<\infty\}$ and are bounded in any compact subset of $\mathfrak{D}$. By a solution of the first boundary value problem for the equation (1) we mean a function $u(x, t)$ which is continuous in $\overline{D}$, bounded with its derivatives appearing in (1) in $\bar{D}$ and satisfies the equation in $\bar{D}-\partial D$ as well as the boundary condition given on ∂D:

$$u|_{\partial D}=\varphi(x,t). \tag{2}$$

§1. A maximum principle. We shall begin with a very simple maximum principle and some of its consequences. The proofs will be omitted.

Theorem 1. *We make the following assumptions on* (1):

i) *The quadratic form* $\sum_{i,j=1}^{n} a_{ij}(x,t,u,p)\xi_i\xi_j$ *is positive definite for every* $(x,t,u,p)\in\mathfrak{D}$ *and for every real vector* ξ;

ii) $f(x,t,u,0)$ *is positive for positive* u.

Then any solution $u(x,t)$ *of* (1) *cannot attain its positive maximum in* $\bar{D}-\partial D$.

Corollary 1. *Let the following assumptions be satisfied:*

i) $\sum_{i,j=1}^{n} a_{ij}(x,t,u,p)\xi_i\xi_j$ *is positive definite;*

ii) $f(x,t,u,0)$ *satisfies one of the following conditions:*

a) $\operatorname{sign} u\cdot f(x,t,u,0)\geqq 0$ *for all* (x,t,u) *under consideration;*

b) $|f(x,t,u,0)|\leqq L|u|$ *with a positive constant* L;

c) $f(x,t,u,0)=f_1(x,t,u,0)+f_2(x,t,u,0)$ *with* $\operatorname{sign} u\cdot f_1(x,t,u,0)\geqq 0$ *and* $|f_2(x,t,u,0)|\leqq L|u|$.

Under these assumptions we conclude that $u(x,t)$ *vanishes identically in* D *if it vanishes on the normal boundary* ∂D *of* D.

Corollary 2.[1] *The first boundary value problem* (1), (2) *has at most one solution, provided that the following conditions are valid:*

i) $a_{ij}(x,t,u,p)$ *satisfy the Lipschitz condition with respect to* u *and the form* $\sum_{i,j=1}^{n} a_{ij}(x,t,u,p)\xi_i\xi_j$ *is positive definite;*

ii) $f(x,t,u,p)$ *is subjected to either of the following:*

a) $f(x,t,u,0)$ *is non-decreasing with respect to* u;

b) $f(x,t,u,p)$ *satisfies the Lipschitz condition with respect to* u.

Corollary 3. *Under the same assumptions as in Corollary* 2 *the Harnack type convergence theorem holds. Let indeed* $u_n(x,t)$ *be a sequence of solutions of the equation* (1) *defined in the domain* D. *Then, it converges uniformly in the whole domain* D *if it converges uniformly on the normal boundary* ∂D *of* D.

§2. A strong maximum principle. In this section we shall generalize the "strong maximum principle" of Nirenberg to the case of quasi-linear parabolic equation (1). Following Nirenberg we begin with a more general equation of the form

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1) A better uniqueness theorem has been obtained by Kaminin and Maslennikova [3].

$$(3)\qquad \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{\lambda,\mu=1}^{m} b_{\lambda\mu}\frac{\partial^2 u}{\partial t_\lambda \partial t_\mu}+\sum_{i=1}^{n} a_i\frac{\partial u}{\partial x_i}+\sum_{\lambda=1}^{m} b_\lambda\frac{\partial u}{\partial t_\lambda}=f$$

where the functions a_{ij}, a_i, $b_{\lambda\mu}$, b_λ, f may depend not only on $x=(x_1,\cdots,x_n)$, $t=(t_1,\cdots,t_m)$ but on u and $\nabla u=(\partial u/\partial x_1,\cdots,\partial u/\partial t_m)$. We assume that all these functions are defined in the domain $\mathcal{G}:\{(x,t)\in G,\ |u|<\infty,\ \|q\|<\infty\}$ (G is a domain in $(n+m)$-space and q is an $(n+m)$-tuple of real numbers), and are bounded in any compact subset in $\mathcal{G}$ and that they are subjected to the following restrictions:

I) There exists a lower semi-continuous function $h(x,t,u,q)>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u,q)\xi_i\xi_j \geqq h(x,t,u,q)\|\xi\|^2$$

for every $(x,t,u,q)\in\mathcal{G}$ and every real vector ξ, while the quadratic form $\sum_{\lambda,\mu=1}^{m} b_{\lambda\mu}(x,t,u,q)\eta_\lambda\eta_\mu$ is positive semi-definite;

II) $f(x,t,u,0)\geqq 0$ for $u\geqq 0$;

III) $f(x,t,u,q)$ satisfies locally the Lipschitz condition with respect to u and q; that is, we have

$$|f(x,t,u,q)-f(x,t,u',q')|\leqq K(|u-u'|+\|q-q'\|)$$

with a positive $K=K(\Omega,M,N)$ depending only on Ω, M and N whenever (x,t) varies in any compact subset Ω and $|u|,|u'|\leqq M$ and $\|q\|$, $\|q'\|\leqq N$.

Theorem 2. *Let the assumptions mentioned above be fulfilled and let a solution $u(x,t)$ of (3) attain its non-negative maximum in G at some interior point P_0 of G. Then we have $u(x,t)\equiv u(P_0)$ in the component in G of the hyperplanes $t_\lambda=const.$ $(\lambda=1,\cdots,m)$ through the point P_0.*

For simplicity it is enough to deal with the case $n=m=1$:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial t^2}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial t}=f,\ A\geqq h>0,\ B\geqq 0.$$

The proof is essentially based on the following lemma.

Lemma. *Let C be a closed circle contained in G and let $P_1=(x_1,t_1)$ be a point on its circumference where the solution $u(x,t)$ achieves its non-negative maximum. Then, the abscissa x_1 of P_1 is equal to that of center of the circle C.*

Proof of Lemma. We may assume that P_1 is the only one maximum point on the circumference of C and that the origin of the coordinate system is situated at the center of C. Let R be the radius of C. Assuming $x_1\neq 0$ we derive an absurdity. We draw a circle C_1 with center P_1 and radius less than R. Define the function $v(x,t)$ by

$$v(x,t)=u(x,t)+\varepsilon v_k(x,t),$$
$$v_k(x,t)=\exp(-k(x^2+t^2))-\exp(-kR^2)$$

where $\varepsilon>0$, $k>0$ are constants. As is easily seen, if ε is sufficiently small $v(x,t)$ is less than or equal to $u(P_1)$ on the boundary of $C\cap C_1$. Hence, $\max_{C\cap C_1} v(x,t)\geqq u(P_1)$ is attained at some point $P_k(x_k,t_k)$ of $C\cap C_1$. Applying the linear parabolic operator

$$\mathscr{P}=A(x,t,u(x,t),\nabla u(x,t))\frac{\partial^2}{\partial x^2}+B(x,t,u,\nabla u)\frac{\partial^2}{\partial t^2}+a\frac{\partial}{\partial x}+b\frac{\partial}{\partial t}$$

to the function v we have, on the one hand, $\mathscr{P}v(P_k)\leqq 0$ in view of the fact that v assumes its maximum at P_k. On the other hand, it follows from the assumptions of the lemma and the relations $|u(P_1)-u(P_k)|\leqq\varepsilon\exp(-k(x_k^2+t_k^2))$, $||\nabla u(P_k)||=2k\varepsilon\sqrt{x_k^2+t_k^2}\exp(-k(x_k^2+t_k^2))$ that

$$\begin{aligned}\mathscr{P}v(P_k)&=\mathscr{P}u(P_k)+\varepsilon\mathscr{P}v_k(P_k)=f(P_k,u(P_k),\nabla u(P_k))+\varepsilon\mathscr{P}v_k(P_k)\\&\geqq -K(|u(P_1)-u(P_k)|+||\nabla u(P_k)||)+\varepsilon\mathscr{P}v_k(P_k)\\&\geqq\varepsilon\exp(-k(x_k^2+t_k^2))[4k^2(Ax_k^2+Bt_k^2)\\&\qquad -2k(A+B+ax_k+bt_k)-2k\sqrt{x_k^2+t_k^2}-K].\end{aligned}$$

The coefficient of k^2 in the bracket of the last expression remains bounded away from zero and hence we finally have $\mathscr{P}v(P_k)>0$ for sufficiently large k. This desired absurdity proves our lemma.

Once the lemma has been established, we can proceed in an entirely the same way as in Nirenberg's paper [4] to complete the proof of Theorem 2.

We now turn to the strong maximum principle for the equation (1).

Theorem 3. *Consider the equation* (1) *concerning which the following conditions are satisfied*:

I) *There exists lower semi-continuous function* $h(x,t,u,p)>0$ *such that*

$$\sum_{i,j=1}^{n}a_{ij}(x,t,u,p)\xi_i\xi_j\geqq h(x,t,u,p)||\xi||^2$$

for every $(x,t,u,p)\in\mathfrak{D}$ *and every real vector* ξ;

II) $f(x,t,u,0)\geqq 0$ *for* $u\geqq 0$;

III) $f(x,t,u,p)$ *satisfies locally the Lipschitz condition with respect to* u *and* p.

Let a solution $u(x,t)$ *of* (1) *in* D *assume its positive maximum at some point* P_0 *on the upper basis of the domain* D *and let* $S(P_0)$ *denote the set of points of* D *that may be connected with* P_0 *by a simple curve in* D *along which the* t *coordinates change monotonically. Then* $u(x,t)\equiv u(P_0)$ *in* $S(P_0)$.

It suffices to give the proof for the case $n=1$ and it also suffices to prove the following

Theorem 3′. *Let* $u(x,t)$ *be a solution of the equation*

$$(1')\qquad A(x,t,u,u_x)\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=f(x,t,u,u_x)\quad(A\geqq h>0)$$

defined and continuous in the rectangle $Q:\{|x|\leqq r,\ t_*\leqq t\leqq t_0\}$ *and let* $u(x,t)$ *assume its positive maximum at the center* P_0 *of the top line. Then* $u(x,t)\equiv u(P_0)$ *in* Q.

Proof of Theorem 3′. It follows from Theorem 2 that $u \equiv u(P_0)$ on the top line of Q. Assume for contradiction that there exists a point $P \in Q$ such that $u(P) < u(P_0)$. Under this assumption it is easy to construct a rectangle $R: \{|x| \leqq r_0,\ t_1 \leqq t \leqq t_0\}$, $0 < r_0 < 1$, such that $u(x,t)$ is less than $u(P_0)$ everywhere in $\overline{R}$ except on its top line. We show that this situation is impossible by introducing a function employed by Il'in, Kalashnikov and Oleinik [2]. Define a function $V(x,t)$ by

$$V(x,t) = m - (m_1 - m)(r_0^2 - x^2)^2 \exp(-k(t-t_1)) - u(x,t),$$

where $m = u(P_0)$, m_1 is a non-negative constant such that $u(x,t_1) \leqq m_1 < m$ and k is a positive constant. As we immediately observe $V(x,t)$ is non-negative on the lateral sides and on the lower line of R, while it takes strictly negative values on the top line. Therefore $\min\limits_{R} V(x,t) < 0$. Let $\overline{P}(\overline{x}, \overline{t})$ be a point of minimum of $V(x,t)$. Clearly $u(\overline{x}, \overline{t}) > m_1 \geqq 0$.

If we operate the operator

$$\mathcal{L} = A(x,t,u(x,t),u_x(x,t)) \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$$

on the function $V(x,t)$, we have $\mathcal{L}V(\overline{P}) \geqq 0$. On the other hand,

$$\begin{aligned} V(\overline{P}) = &-(m-m_1)[8A\overline{x}^2 - 4A(r_0^2 - \overline{x}^2)^2 + k(r_0^2 - \overline{x}^2)^2] \exp(-k(\overline{t} - t_1)) \\ &-(f(\overline{P}, u(\overline{P}), u_x(\overline{P})) - f(\overline{P}, u(\overline{P}), 0)) - f(\overline{P}, u(\overline{P}), 0). \end{aligned}$$

Paying attention to the assumptions of the theorem and to an equality $u_x(\overline{P}) = 4(m-m_1)\overline{x}(r_0^2 - \overline{x}^2) \exp(-k(\overline{t} - t_1))$ resulting from $V_x(\overline{P}) = 0$, we finally obtain

$$\begin{aligned} \mathcal{L}V(\overline{P}) \leqq &-(m-m_1)[8A\overline{x}^2 - 4(A + L|\overline{x}|)(r_0^2 - \overline{x}^2) \\ &+ k(r_0^2 - \overline{x}^2)^2] \exp(-k(\overline{t} - t_1)). \end{aligned}$$

The function $8Ax^2 - 4(A + L|x|)(r_0^2 - x^2) + k(r_0^2 - x^2)^2$ can be shown to be made positive in R for sufficiently large with k the aid of the reasoning similar to that presented in [2]. We are thus lead to a contradiction and the theorem is proved.

Corollary 1. Let the following conditions be satisfied:

I) $\sum\limits_{i,j=1}^{n} a_{ij}(x,t,u,p)\xi_i\xi_j \geqq h(x,t,u,p)\,||\xi||^2$;

II) sign $u \cdot f(x,t,u,0) \geqq 0$;

III) $f(x,t,u,p)$ *satisfies locally the Lipschiz condition with respect to* u *and* p.

Then for any solution $u(x,t)$ *of* (1) *continuous in* $\overline{D}$

$$|u(x,t)| \leqq \max_{\partial D} |u(x,t)|, \quad (x,t) \in \overline{D} - \partial D.$$

The equality sign can be removed in case u *is not constant.*

Corollary 2. *Consider the equation*

$$(4) \qquad \sum_{i,j=1}^{n} a_{ij}(x, t, \text{grad } u)\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f(x, t, u, \text{grad } u)$$

under the conditions:

I) *There exists a lower semi-continuous function* $h(x, t, p) > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{ij}(x, t, p)\xi_i\xi_j \geqq h(x, t, p)\,||\xi||^2;$$

II) $a_{ij}(x, t, p)$ *satisfy locally the Lipschitz condition with respect to* ρ;

III) $f(x, t, u, p)$ *is non-decreasing with respect to* u;

IV) $f(x, t, u, p)$ *satisfies locally the Lipschitz condition with respect to* u *and* p.

Under these assumptions we can assert that the boundary value problem (4), (2) *has at most one solution.*

Corollary 3. *Let the same assumptions as in Corollary* 2 *be satisfied with regard to the equation* (4) *and let* $u_n(x, t)$ *be a sequence of solutions defined and continuous in* $\overline{D}$. *Then,* $u_n(x, t)$ *converges uniformly in* D *if it converges uniformly on the normal boundary* ∂D *of* D.

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