48. A Note on a Weak Subsolution

By Kazunari HAYASHIDA
Department of Applied Mathematics, Nagoya University

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1. Let L be an elliptic differential operator of order $2s$ defined in a domain $\mathfrak{D}$ of the euclidean n-space R^n:

$$(1) \qquad L=\sum_{0<|\alpha|\leqq 2s} a_\alpha(x)D^\alpha,\ D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},\ \alpha=(\alpha_1,\cdots,\alpha_n),$$

where $a_\alpha(x)\in C^{|\alpha|}(\mathfrak{D})(|\alpha|=\alpha_1+\cdots+\alpha_n)$. If a measurable function u is essentially bounded from the above in $\mathfrak{D}$ and satisfies the inequality

$$\int_{\mathfrak{D}} u(x)L^*\varphi(x)dx\geqq 0$$

for all non-negative functions $\varphi\in C^{2s}(\mathfrak{D})$ with compact carrier in $\mathfrak{D}$, where L^* is the adjoint operator of L, then we say that u is a weak L-subsolution in $\mathfrak{D}$. In the case when L is of second order, a weak L-subsolution is a weakly L-subharmonic function in the sense of Littman [2]. In this note, we shall prove the following

Theorem. *If u is a weak L-subsolution in $\mathfrak{D}$ and assumes its essential supremum M (over $\mathfrak{D}$) almost everywhere in an open set in $\mathfrak{D}$, then $u=M$ almost everywhere in $\mathfrak{D}$.*

This theorem for a weakly L-subharmonic function u was proved by Littman (Theorem 2 in [2]).

2. We prepare some lemmas. Consider the function

$$\phi_{R_0}(R)=\begin{cases}0 & \text{for } R\leqq 0,\\ e^{-\frac{1}{R}}e^{-\frac{1}{R_0-R}} & \text{for } 0<R<R_0,\\ 0 & \text{for } R_0\leqq R.\end{cases}$$

Clearly $\phi_{R_0}(R)$ is an infinitely differentiable function with compact carrier in $(-\infty,\infty)$.

Lemma 1. *For an arbitrary positive integer h, there exists a positive number δ_h such that, if $0<R_0-R<\delta_h$,*

$$\phi_{R_0}^{(h)}(R)=(-1)^h|\phi_{R_0}^{(h)}(R)|,$$

where $\phi_{R_0}^{(h)}(R)=\dfrac{d^h}{dR^h}\phi_{R_0}(R)$.

Proof. We prove the lemma by induction on h. Our lemma is obvious for $h=0$. Assume the assertion for $h=k$. We see easily that $\phi_{R_0}^{(k+1)}(R)$ can be written in the form

$$\phi_{R_0}^{(k+1)}(R)=\frac{Q_k(R)}{P_k(R)}\phi_{R_0}(R).$$

Here $P_k(R)$ and $Q_k(R)$ are both polynomials with respect to a variable

R. In addition $P_k(R)$ has no zero except $R=0$ and $R=R_0$. If we take a positive number $\delta(<\delta_k)$ sufficiently small, $Q_k(R)$ and $\phi_{R_0}^{(k+1)}(R)$ have a definite sign in $0<R_0-R<\delta$. And by the mean value theorem, we can find R' such that

$$\phi_{R_0}^{(k)}(R)=(R-R_0)\phi_{R_0}^{(k+1)}(R'),\ (R<R'<R_0,\ 0<R_0-R<\delta).$$

Since from our assumption the sign of the left hand side in this equality is $(-1)^k$ in $0<R_0-R<\delta_k$, the sign of $\phi_{R_0}^{(k+1)}(R')$ is $(-1)^{k+1}$. Hence, putting $\delta_{k+1}=\delta$, we obtain the required.

Hereafter, we assume that the origin 0 of R^n is in $\mathfrak{D}$. We put

$$W(x)=\int_{r^2}^{\infty}\left(\frac{1}{r^{2k}}-\frac{1}{R^k}\right)\phi_{R_0}(R)dR,\quad r=|x| =\sqrt{x_1^2+\cdots+x_n^2}, \tag{2}$$

where k is positive and $\sqrt{R_0}$ (<1) is smaller than the distance of the boundary of $\mathfrak{D}$ from the origin.

Lemma 2. *Let L^* be the adjoint operator of L. Then there exists a constant k_0 depending only on R_0 and on L such that if $k\geqq k_0$, it holds that*

$$L^*W(x)>0 \quad \textit{in } 0<R_0-r^2<\delta_0,$$

where $\delta_0=\underset{0\leqq h\leqq 2s}{\mathrm{Min}}\,\delta_h$ and δ_h $(h=1,2,\cdots)$ are those in Lemma 1.

Proof. By putting $\rho=r^2$ and $f(\rho)=W(x)=\int_{\rho}^{\infty}\left(\frac{1}{\rho^k}-\frac{1}{R^k}\right)\phi_{R_0}(R)$ dR, we see

$$\frac{d}{d\rho}f(\rho)=\frac{-k}{\rho^{k+1}}\int_{\rho}^{\infty}\phi_{R_0}(R)dR. \tag{3}$$

Applying the Leibniz formula to (3), we have

$$\begin{aligned}\frac{d^m}{d\rho^m}f(\rho)&=\sum_{l=0}^{m-1}\binom{m-1}{l}\left(\frac{d}{d\rho}\right)^l\left(\frac{-k}{\rho^{k+1}}\right)\left(\frac{d}{d\rho}\right)^{m-1-l}\int_{\rho}^{\infty}\phi_{R_0}(R)dR\\&=\sum_{l=0}^{m-1}\binom{m-1}{l}\frac{(-1)^{l+1}k(k+1)\cdots(k+l)}{\rho^{k+l+1}}\{-\phi_{R_0}^{(m-2-l)}(\rho)\},\\&\qquad\qquad(1\leqq m\leqq 2s),\end{aligned} \tag{4}$$

where $\phi_{R_0}^{(-1)}(\rho)=-\int_{\rho}^{\infty}\phi_{R_0}(R)dR(\leqq 0)$. By Lemma 1, we have

$$\phi_{R_0}^{(m-2-l)}(\rho)=(-1)^{m-l}|\phi_{R_0}^{(m-2-l)}(\rho)|,\quad (0\leqq m-2-l) \tag{5}$$

in $0<R_0-\rho<\delta_0$. Substituting (5) into the right hand side of (4), we obtain

$$\frac{d^m}{d\rho^m}f(\rho)=(-1)^m\sum_{l=0}^{m-1}\binom{m-1}{l}\frac{k(k+1)\cdots(k+l)}{\rho^{k+l+1}}|\phi_{R_0}^{(m-2-l)}(\rho)|,\quad (1\leqq m\leqq 2s). \tag{6}$$

Putting $m=2s$ in (6), we get

$$\left|\frac{d^{2s}}{d\rho^{2s}}f(\rho)\right|\geqq\sum_{l=0}^{2s-1}\frac{k(k+1)\cdots(k+l)}{\rho^{k+l+1}}|\phi_{R_0}^{(2s-2-l)}(\rho)|,\quad 0<R_0-\rho<\delta_0. \tag{7}$$

And, in general,

$$(8)\qquad \left|\frac{d^m}{d\rho^m}f(\rho)\right| \leqq A_m \sum_{l=0}^{m-1} \frac{k(k+1)\cdots(k+l)}{\rho^{k+l+1}} |\phi_{R_0}^{(m-2-l)}(\rho)|, \quad 0<R_0-\rho<\delta_0,$$

where A_m is a constant depending only on m.

Now computing $D^\alpha W(x)$ similarly as in [1], we have

$$(9)\qquad D^\alpha W(x)=\sum_{q=1}^{|\alpha|}\alpha_1!\cdots\alpha_n!\frac{d^q}{d\rho^q}f(\rho)\cdot\left(\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}\cdot x_1^{\beta_1}\cdots x_n^{\beta_n}\right).$$

By setting

$$L^*=\sum_{|\alpha|\leqq 2s} b_\alpha(x)D^\alpha,$$

it holds that

$$(10)\qquad \begin{aligned} L^*W(x)=&\sum_{|\alpha|=2s} b_\alpha(x)\alpha_1!\cdots\alpha_n!\frac{d^{|\alpha|}}{d\rho^{|\alpha|}}f(\rho)\cdot\frac{2^{|\alpha|}}{\alpha_1!\cdots\alpha_n!}\cdot x_1^{\alpha_1}\cdots x_n^{\alpha_n}\\ &+\sum_{|\alpha|=2s} b_\alpha(x)\alpha_1!\cdots\alpha_n!\sum_{q=1}^{|\alpha|-1}\frac{d^q}{d\rho^q}f(\rho)\cdot\left(\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}\cdot x_1^{\beta_1}\cdots x_n^{\beta_n}\right)\\ &+\sum_{|\alpha|\leqq 2s-1} b_\alpha(x)\alpha_1!\cdots\alpha_n!\sum_{q=1}^{|\alpha|}\frac{d^q}{d\rho^q}f(\rho)\cdot\left(\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}\cdot x_1^{\beta_1}\cdots x_n^{\beta_n}\right). \end{aligned}$$

By ellipticity of L^*, there is a positive constant c such that, if $r^2=\rho\leqq R_0$,

$$(11)\qquad \sum_{|\alpha|=2s} b_\alpha(x)x_1^{\alpha_1}\cdots x_n^{\alpha_n}\geqq cr^{2s}.$$

Substituting (11) into the first term on the right hand side of (10), we see, in $0<R_0-\delta_0<\rho$,

$$\begin{aligned} L^*W(x)\geqq\,& 2^{2s}c\frac{d^{2s}}{d\rho^{2s}}f(\rho)r^{2s}\\ &-M_1(2s+1)^n(2s)!\sum_{q=s}^{2s-1}\frac{d^q}{d\rho^q}f(\rho)\cdot\frac{(2n)^{2q-2s}}{(2q-2s)!}r^{2q-2s}\\ &-M_2\sum_{|\alpha|\leqq 2s-1}(|\alpha|+1)^n|\alpha|!\sum_{q=\left[\frac{|\alpha|}{2}\right]+1}^{|\alpha|}\frac{d^q}{d\rho^q}f(\rho)\cdot\frac{(2n)^{2q-|\alpha|}}{(2q-|\alpha|)!}r^{2q-|\alpha|}, \end{aligned}$$

where $M_1=\sup_{|\alpha|=2s}|b_\alpha(x)|$ and $M_2=\sup_{|\alpha|\leqq 2s-1}|b_\alpha(x)|$. If we put $B_0=2^{2s}c$, $B_m=M_1(2s+1)^n(2s)!\dfrac{(2n)^{2m-2s}}{(2m-2s)!}A_m$ and $B_{\alpha,q}=M_2(|\alpha|+1)^n|\alpha|!\dfrac{(2n)^{2q-|\alpha|}}{(2q-|\alpha|)!}\cdot A_q$, then from (7) and (8), we have

$$\begin{aligned} L^*W(x)\geqq\,& B_0\left(\sum_{l=0}^{2s-1}\frac{k(k+1)\cdots(k+l)}{\rho^{k+l+1-s}}|\phi^{(2s-l-2)}(\rho)|\right)\\ &-\sum_{q=s}^{2s-1}B_q\left(\sum_{l'=0}^{q-1}\frac{k(k+1)\cdots(k+l')}{\rho^{k+l'+1-q+s}}|\phi^{(q-l'-2)}(\rho)|\right)\\ &-\sum_{|\alpha|\leqq 2s-1}\sum_{q=\left[\frac{|\alpha|}{2}\right]+1}^{|\alpha|}B_{\alpha,q}\left(\sum_{l''=0}^{q-1}\frac{k(k+1)\cdots(k+l'')}{\rho^{k+l''+1-q+\frac{|\alpha|}{2}}}\phi^{(q-l''-2)}(\rho)|\right), \end{aligned}$$

in $0<R_0-\delta_0<\rho$. On the right hand side of this inequality we compare the coefficient of $|\phi^{(2s-l-2)}|$ in the first sum with the coefficient of $|\phi^{(q-l''-2)}|$ in the third sum. If $2s-l-2=q-l''-2$, then $k+l>k+l''$ and $k+l+1-s>k+l''+1-q+\frac{|\alpha|}{2}$. Therefore, we can take k_0' such that, when $k\geqq k_0'$

$$L^*W(x)\geqq\frac{B_0}{2}\Big(\sum_{l=0}^{2s-1}\frac{k(k+1)\cdots(k+l)}{\rho^{k+l+1-s}}|\phi^{(2s-l-2)}(\rho)|-\sum_{q=s}^{2s-1}B_q\Big(\sum_{l'=0}^{q-1}\frac{k(k+1)\cdots(k+l')}{\rho^{k+l'+1-q+s}}|\phi^{(q-l'-2)}(\rho)|\Big),\quad (0<R_0-\delta_0<\rho). \tag{12}$$

In the same manner as above, we compare the first sum with the second sum on the right hand side of (12). Let $2s-l-2=q-l'-2$, then $k+l+1-s=k+l'+1-q+s$ and $k+l>k+l'$. Hence we can take suitably k_0 for which our lemma holds.

3. Proof of the theorem. Denote by S the maximal open set in which $u=M$ almost everywhere. We assume that $S\neq\mathfrak{D}$. As is easily seen, if we take R_0 sufficiently small, there exist concentric spheres E_1 and E_2 satisfying the conditions:

i) the radius of E_2 and the radius of E_1 are $\sqrt{R_0}$ and $\sqrt{R_0-\delta_0}$ respectively (δ_0 is that in Lemma 2).

ii) E_1 lies in S and $\overline{E}_2$ lies in $\mathfrak{D}$.

iii) $\overline{E}_1$ contains boundary point P of S which belongs to $\mathfrak{D}$.

We may assume that the center of E_1 is the origin. We construct a non-negative infinitely differentiable function $w(x)$ which equals $W(x)$ in E_1^c. Since u is a weak L-subsolusion in $\mathfrak{D}$, it holds

$$0\geqq\int_{\mathfrak{D}}(M-u)L^*w\,dx. \tag{13}$$

On the other hand we have

$$\int_{\mathfrak{D}}(M-u)L^*w\,dx=\int_{S'}(M-u)L^*w\,dx+\int_{S}(M-u)L^*w\,dx+\int_{\mathfrak{D}-(S\cup S')}(M-u)L^*w\,dx, \tag{14}$$

where $S'=S^c\frown E_2$. On the right hand side of (14), the second term vanishes and the last term also vanishes, as $w=W=0$ in E_2^c. Hence from (13), we have

$$0\geqq\int_{S'}(M-u)L^*w\,dx=\int_{S'}(M-u)L^*W\,dx.$$

This inequality implies $M-u=0$ almost everywhere in E_2-E_1. That is, $u=M$ almost everywhere in a neighborhood of P, which is a contradiction. Hence S is identical with $\mathfrak{D}$. Thus our theorem is proved.

References

[1] K. Hayashida: Unique continuation theorem of elliptic systems of partial differential equations, Proc. Japan Acad., **38**, 630–635 (1962).

[2] W. Littman: A strong maximum principle for weakly L-subharmonic functions, Journ. of Math. Mech., **8**, 761–770 (1959).