5. On a Boundary Theorem on Open Riemann Surfaces

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1. **Introduction.** Let U be the class of Riemann surfaces on which there exist the Green function and at least a bounded minimal positive harmonic function (C. Constantinescu and A. Cornea [1]) and O_L be the class of Riemann surfaces on which there exist the Green function and no non-constant Lindelöfian meromorphic function (M. Heins [3]). Let R be an open Riemann surface and Ω be a subregion of the Riemann surface R whose relative boundary $\partial\Omega$ with respect to R consists of at most an enumerable number of analytic curves clustering nowhere in R. If there exists no non-constant single-valued bounded harmonic function in Ω which vanishes continuously on $\partial\Omega$, we say that Ω belongs to SO_{HB}. The following theorem was proved by many authors (see [2], [4], [6], and [8]).

Let R be an open Riemann surface belonging to the class U and Ω be a subregion of R which satisfies the above boundary condition and does not belong to SO_{HB}, then Ω belongs to O_L.

In the present paper we shall give another simple proof of this assertion with aid of the notion of thinness in Martin's space [5] (which is given by Martin's compactification of an open Riemann surface), introduced by L. Naïm [7].

2. **Preliminaries.** We shall introduce the notion of thinness and some useful results for our purpose.

Let R be an open Riemann surface and $\hat{R}$ be Martin's space associated with R. We say that $\Delta^R=\hat{R}-R$ is the Martin boundary of R. Now let $K_x(y)$ be a kernel function in the sense of Martin, that is $K_x(y)=\dfrac{G(x,y)}{G(x,y_0)}$ for $x\in\hat{R}-\{y_0\}$, $y\in R$ with a fixed point y_0 in R. Then x_0 is said to be a minimal point of Δ^R if $K_{x_0}(y)$ is a minimal positive harmonic function in R in the sense of Martin and x_0 is said to be a bounded minimal point of Δ^R if, in addition, $K_{x_0}(y)$ is bounded in R.

Let m be a positive measure in R, then a K-potential with respect to the measure m in R is defined in $\hat{R}-\{y_0\}$ by

$$U(x)=\int K_x(y)\,dm(y).$$

Definition. *A subset E of R is said to be thin at a point x_0 in*

$\hat{R}-\{y_0\}$ *if* x_0 *is not a limit point of* E *or otherwise if there is a K-potential such that*

$$U(x_0)<\liminf_{\substack{x\to x_0\ x\in E\\ x\neq x_0}} U(x).$$

Then we can immediately see that the union of a finite number of the thin sets at x_0 is also thin.

Naïm [7] proved the following:

(2.1) R is not thin at any minimal point x_0 of $\varDelta^R$ and vice versa (Theorem 3).

(2.2) A set E of R is thin at a minimal point x_0, if and only if the extremization $\mathcal{E}_{K_{x_0}}^{R-E}$ of the kernel function $K_{x_0}(y)$ over $R-E$ does not conserve this function, that is,

$$\mathcal{E}_{K_{x_0}}^{R-E}(y) \neq K_{x_0}(y) \quad \text{(Theorem 5)}.$$

Here the notion of the extremization is the following:

The extremization $\mathcal{E}_v^E$ *of the positive superharmonic function* v *over the set* E *is the least positive superharmonic function which dominates* v *in* $R-E$ *except for a set of capacity zero.*

(2.3) Let u be a harmonic function in R, Ω be an open set of R and $\overset{*}{\Omega}$ be a boundary of Ω with respect to Martin's space R. Let $\underset{*}{u}$ be the function on $\overset{*}{\Omega}$ which coincides with u on $\overset{*}{\Omega}\cap R$ and 0 on $\overset{*}{\Omega}\cap\varDelta^R$ and $H_{\underset{*}{u}}^{\Omega}(y)$ be the solution of Dirichlet problem with respect to Ω in the sense of Brelot.

Let x_0 be a point of $\overset{*}{\Omega}$ being minimal in $\varDelta^R$. If $u=K_{x_0}(y)$ is different from $H_{\underset{*}{u}}^{\Omega}(y)$, then the difference $u(y)-H_{\underset{*}{u}}^{\Omega}(y)$ is a minimal positive harmonic function in Ω (Theorem 12).

On the other hand, Heins [3] proved the following assertion:

Let f be a single-valued meromorphic function in R, and $\mathfrak{E}$ be a subset of the w-sphere. For each open set δ of the w-sphere, we shall denote the greatest harmonic minorant of the extremization of the constant 1 over $R-f^{-1}(\delta)$ by $\hat{\mathcal{E}}_1^{R-f^{-1}(\delta)}(y)$ and the lower envelope of the family $\{\hat{\mathcal{E}}_1^{R-f^{-1}(\delta)}(y)\}_{\delta\supset}$ by $B_{\mathfrak{E}}$.

(2.4) If f is Lindelöfian, then Cap $\mathfrak{E}=0$ implies $B_{\mathfrak{E}}=0$.

3. **Theorems.** Using these results we shall prove the following theorem:

Theorem 1. *Let* R *be an open Riemann surface belonging to the class* U, *then* R *belongs to the class* O_L.

Proof. Suppose that there exists an open Riemann surface R which belongs to the class U and does not belong to the class O_L. Let f be a non constant Lindelöfian meromorphic function in R and x_0 be a bounded minimal point of the Martin boundary $\varDelta^R$.

On the other hand we can consider as $\mathfrak{E}$ a single point w of the

w-sphere and as δ an open neighborhood $V(w)$ of the w, so $B_{\mathfrak{F}}$ coincides with the lower envelope of the family $\{\hat{\mathcal{E}}_1^{R-f^{-1}(V(w))}(y)\}$.

Now we see that

$$\mathcal{E}_1^{R-f^{-1}(V(w))}(y) \geq k \cdot \mathcal{E}_{K_{x_0}}^{R-f^{-1}(V(w))}(y),$$

where $k = 1/\sup_R K_{x_0}(y) > 0$, since $K_{x_0}(y)$ is bounded in R.

Then there exists a small neighborhood $V(w)$ of w such that

$$\mathcal{E}_{K_{x_0}}^{R-f^{-1}(V(w))}(y) \not\equiv K_{x_0}(y),$$

therefore $f^{-1}(V(w))$ is thin x_0 by (2.2).

In fact if we assume that for any $V(w)$

$$\mathcal{E}_{K_{x_0}}^{R-f^{-1}(V(w))}(y) \equiv K_{x_0}(y)$$

by the definition of the greatest harmonic minorant, we have

$$\hat{\mathcal{E}}_1^{R-f^{-1}(V(w))}(y) \geq k \cdot \hat{\mathcal{E}}_{K_{x_0}}^{R-f^{-1}(V(w))} \equiv k \cdot K_{x_0}(y)$$

and $\qquad \hat{\mathcal{E}}_1^{R-f^{-1}(V(w))}(y_0) \geq k \cdot K_{x_0}(y_0) = k > 0$ for any $V(w)$.

For a small positive number ε $(<k)$ there exists a small $V(w)$ such that $\qquad \hat{\mathcal{E}}_1^{R-f^{-1}(V(w))}(y_0) < \varepsilon,$

since $B_{\{w\}} = 0$ by (2.4). This is impossible.

Thus for any point w of the w-sphere we can choose an open neighborhood $V(w)$ of w such that $f^{-1}(V(w))$ is thin at x_0.

The family $\{V(w)\}_{w \in w\text{-sphere}}$ is an open covering of the w-sphere and we can choose a finite number of $V(w_i)$ $(i=1, \cdots, n)$ such that $\{V(w_i)\}_{i=1}^n$ is a covering of the w-sphere by the compactness of this.

Every $f^{-1}(V(w_i))$ is thin at x_0, so $\bigcup_{i=1}^n f^{-1}(V(w_i))$ is also thin at the point x_0 of Δ^R. But this set coincides with R. This contradicts (2.1) and leads to our assertion.

As a consequence of Theorem 1 we have

Theorem 2. *Let R be an open Riemann surface belonging to the class U and Ω be a subregion of R such that $R-\Omega$ is thin at some bounded minimal point x_0 of the Martin boundary Δ^R, then Ω belongs to the class O_L.*

Proof. We know that $H_{\underset{*}{K_{x_0}}}^{\Omega}(y) \equiv \mathcal{E}_{K_{x_0}}^{\Omega}(y)$ in Ω. Since $R-\Omega$ is thin at x_0, $\mathcal{E}_{K_{x_0}}^{\Omega}(y) \not\equiv K_{x_0}(y)$.

Then by the property of the extremization we see that $K_{x_0}(y) - H_{\underset{*}{K_{x_0}}}^{\Omega}(y) > 0$ in Ω, and by (2.3) $K_{x_0}(y) - H_{\underset{*}{K_{x_0}}}^{\Omega}(y) > 0$ is a bounded minimal harmonic function in Ω. This shows us that Ω belongs to the class U. We conclude by Theorem 1 that Ω belongs to the class O_L.

References

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