100. On Pseudocompactness and Continuous Mappings

By Sitiro HANAI and Akihiro OKUYAMA
Osaka University of Liberal Arts and Educations

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Let X and Y be completely regular T_1-spaces and let φ be a continuous mapping of X onto Y.

M. Henriksen and J. R. Isbell have shown that the following proposition is not true by a counter example. *If φ is a fitting map*[1] *and if Y is pseudo compact, then X is pseudocompact.* (Cf. [3], p. 93.)

Now we shall show that if φ is an open Z-mapping[2] of X onto Y and if Y is pseudocompact, then X is pseudocompact. As an immediate consequence of this fact we have a theorem concerning the pseudocompactness of the product space which was shown in [1].

Theorem 1. *Let $\varphi(X)=Y$ be an open Z-mapping such that for each point y of Y $\varphi^{-1}(y)$ is relatively pseudocompact.*[3] *If Y is pseudocompact, then X is pseudocompact.*

Proof. Suppose that X is not pseudocompact. Then there exists a positive unbounded continuous function f on X such that $f^{-1}(n)$ is not empty for each positive integer n. Let x_n be a point of $f^{-1}(n)$. Since $\varphi^{-1}(y)$ is relatively pseudocompact, we can, without loss of generality, assume that for any two distinct integrs m, n $\varphi(x_m)\neq\varphi(x_n)$.

Let $U_n=\left\{x\in X; |f(x)-n|<\dfrac{1}{4}\right\}$ for each $n\ (\geqq 2)$. Hence we shall show that for any subcollection $\{\overline{U}_{ni}; i=1,2,\cdots\}$ of $\{\overline{U}_n; n=1,2,\cdots\}$ the set $\bigcup_{i=1}^{\infty}\overline{U}_{ni}$ is a zero-set, where $n_i>n_j\ (i<j)$.

For any two distinct integers $m, n\ (1<m<n)$ we define a function f_{mn} on closed interval $[m, n]$ (in real line) as follows:

$$f_{mn}(r)=\begin{cases} (r-m)\vee\dfrac{1}{4}-\dfrac{1}{4} & \left(m\leqq r\leqq\dfrac{m+n}{2}\right) \\ (n-r)\vee\dfrac{1}{4}-\dfrac{1}{4} & (\quad r\leqq n). \end{cases}$$

where $a\vee b$ denotes the maximum of a and b. Then f_{mn} is continuous.

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1) A closed continuous mapping φ of a space X onto a space Y such that for each point $y\in Y$, the set $\varphi^{-1}(y)$ is compact, is called a *fitting map*. (Cf. [3] p. 84.)

2) A mapping φ of X onto Y is called *Z-mapping* if every zero-set $Z(f)=\{x; f(x)=0\}$, $f\in C(X)$, is mapped to a closed subset of Y. (Cf. [2] p. 119.)

3) A subset F of a space X is said to be *relatively pseudocompact* if every continuous function on X is bounded on F. (Cf. [4].)

Next, we define a function g on X as follows:

$$g(x)=\begin{cases} \dfrac{n_1-1}{2}-\dfrac{1}{4} & \left(0<f(x)\leqq\dfrac{1+n_1}{2}\right) \\ f_{1n_1}(f(x)) & \left(\dfrac{1+n_1}{2}\leqq f(x)\leqq n_1\right) \\ f_{n_in_{i+1}}(f(x)) & (n_i\leqq f(x)\leqq n_{i+1})\ (i=1,2,\cdots). \end{cases}$$

Then g is continuous. In fact, (i) if $f(x)=\dfrac{1+n_1}{2}$, then $f_{1n_1}(f(x))=f_{1n_1}\left(\dfrac{1+n_1}{2}\right)=\left(\dfrac{1+n_1}{2}-1\right)^{\vee}\dfrac{1}{4}-\dfrac{1}{4}=\dfrac{n_1-1}{2}-\dfrac{1}{4}$, (ii) if $f(x)=n_i$, then $f_{n_{i-1}n_i}(f(x))=f_{n_{i-1}n_i}(n_i)=(n_i-n_i)^{\vee}\dfrac{1}{4}-\dfrac{1}{4}=f_{n_in_{i+1}}(fx))$ for all $i=1,2,\cdots$ where $n_0=1$. Since f is continuous, g is continuous.

Now we shall show $g^{-1}(0)=\bigcup_{i=1}^{\infty}\overline{U}_{n_i}$. If x is an arbitrary point of $\bigcup_{i=1}^{\infty}\overline{U}_{n_i}$, then there is some k such as $x\in\overline{U}_{n_k}$. From the definition of $\overline{U}_{n_k}$ we have $|n_k-f(x)|\leqq\dfrac{1}{4}$. To show that $g(x)=0$, it is sufficient to consider the following two cases.

Case 1). $0\leqq n_k-f(x)\leqq\dfrac{1}{4}$. If $k=1$, then

$$g(x)=f_{1n_1}(f(x)=(n_1-f(x))^{\vee}\frac{1}{4}-\frac{1}{4}=0.$$

If $k>1$, then

$$g(x)=f_{n_{k-1}n_k}(f(x))=(n_k-f(x))^{\vee}\frac{1}{4}-\frac{1}{4}=0.$$

Case 2). $0\leqq f(x)-n_k\leqq\dfrac{1}{4}$. In this case

$$g(x)=f_{n_kn_{k+1}}(f(x))=(f(x)-n_k)^{\vee}\frac{1}{4}-\frac{1}{4}=0.$$

Therefore, we have $g(x)=0$ in both cases and, consequently, $g^{-1}(0)\supset\bigcup_{i=1}^{\infty}\overline{U}_{n_i}$. Conversely, let x be an arbitrary point of $g^{-1}(0)$. If $\dfrac{1+n_1}{2}\leqq f(x)\leqq n_1$, then we have

$$0=g(x)=f_{1n_1}(f(x))=(n_1-f(x))^{\vee}\frac{1}{4}-\frac{1}{4},$$

and, therefore, we have $x\in\overline{U}_{n_1}$. If $n_i\leqq f(x)\leqq\dfrac{n_i+n_{i+1}}{2}$ then

$$0=g(x)=f_{n_in_{i+1}}(f(x))=(f(x)-n_i)^{\vee}\frac{1}{4}-\frac{1}{4}.$$ Thus we have $x\in\overline{U}_{n_i}$. If $\dfrac{n_i+n_{i+1}}{2}\leqq f(x)\leqq n_{i+1}$, then

$$0=g(x)=f_{n_in_{i+1}}(f(x))=(n_{i+1}-f(x))^{\vee}\frac{1}{4}-\frac{1}{4}.$$

Then we have $x\in\overline{U}_{n_{i+1}}$. In all cases we have $x\in\bigcup_{i=1}^{\infty}\overline{U}_{n_i}$ and, hence,

$g^{-1}(0)\subset\bigcup_{i=1}^{\infty}\overline{U}_{n_i}$. Therefore, $\bigcup_{i=1}^{\infty}\overline{U}_{n_i}$ is a zero-set.

Since φ is Z-mapping, for any subcollection $\{U_{n_i};\ i=1,2,\cdots\}$ of $\{U_n;\ n=1,2,\cdots\}$ $\varphi(\bigcup_{i=1}^{\infty}\overline{U}_{n_i})=\bigcup_{i=1}^{\infty}\varphi(\overline{U}_{n_i})$ is closed in Y and, in particular, $\bigcup_{n=1}^{\infty}\varphi(\overline{U}_n)$ is closed in Y.

From the assumption that $\varphi^{-1}(y)$ is relatively pseudocompact for each $y(\in Y)$, y is contained in only a finite number of $\{\varphi(\overline{U}_n);\ n=1,2,\cdots\}$. Thus $\{\varphi(\overline{U}_n);\ n=1,2,\cdots\}$ is a locally finite collection of closed sets of Y. That is, for any point y of $\bigcup_{n=1}^{\infty}\varphi(\overline{U}_n)$ the neighborhood $U=Y-\bigcup\{\varphi(\overline{U}_{n_i});\ \varphi(\overline{U}_{n_i})\not\ni y\}$ of y (in Y) intersects only a finite number of $\{\varphi(\overline{U}_n);\ n=1,2,\cdots\}$. Since φ is an open mapping and $\varphi^{-1}(y)$ is relatively pseudocompact for each $y(\in Y)$, $\{\varphi(U_n);\ n=1,2,\cdots\}$ is an infinite, locally finite collection of open sets of Y. But this contradicts the assumption that Y is pseudocompact ([1], Theorem 3). This completes the proof of the theorem.

Remarks 1. In our theorem, if we omit the assumption that $\varphi^{-1}(y)$ is relatively pseudocompact, then it is not true. For example, if X is a countable discrete space, Y is a single point, and if $\varphi(X)=Y$ is a constant map, then X is not pseudocompact, though φ is an open Z-mapping and Y is pseudocompact.

2. In our theorem we cannot omit the assumption that φ is an open mapping. (Cf. [3], p. 93.)

3. The following example shows that the assumption that φ is a Z-mapping is necessary in our theorem.

Let X be a subspace of Euclidean plane such that $\{(x,x');\ 0\leqq x<1,\ 0\leqq x'\leqq 1\}\cup\{(1,0)\}$ and let $Y=[0,1]$ be a closed interval of real line. If $\varphi(X)=Y$ is a mapping such that for any point (x,x') of $X\varphi((x,x'))=x$, then φ is an open continuous mapping. But φ is not Z-mapping. For, if we put a subset $A=\{(x,x);\ 0\leqq x<1\}$ of X, then $\varphi(A)=[0,1)$ is not closed in Y, although A is a zero-set. Since Y is compact, Y is pseudocompact. Let $U_n=\left\{(x,x')\in X;\ \left|x-\frac{1}{2^n}\right|<\frac{1}{2^{n+1}},\ \frac{1}{2}<x'\leqq 1\right\}$. Then the collection $\{U_n;\ n=1,2,\cdots\}$ is locally finite in X. This means that X is not pseudocompact. (Cf. [1], Theorem 3).

T. Isiwata has proved that *X is pseudocompact if and only if the projection $Y\times X\to Y$ is a Z-mapping for some weakly separable space Y*. (Cf. [4].)

Using the above fact and our theorem, we have immediately the following Theorem 2.

Theorem 2. (*Bargley, Connell and Mcnight*) *If X is a weakly separable space, then the topological product $X\times Y$ of X and Y is* *pseudocompact if and only if both X and Y are pseudocompact.*

References

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