# 129. On the Propagation of Regularity of Solutions of Partial Differential Equations with Constant Coefficients

By Taira SHIROTA

Department of Mathematics, Osaka University

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1. Let $P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$ be a partial differential operator of order $m$ with constant coefficients. Let $\xi$ be a unit vector of the dual space $\Xi^n$ of $R^n=\{(x_1, x_2, \cdots, x_n)\}$ and for any vector $\xi$, $S(\xi, h)$ the spherical neighbourhood of $\xi$ with radius $h$. Then we define the $\xi$-regularity of $P$ as follows:

**Definition.** $P\left(\frac{\partial}{\partial x}\right)$ is $\xi$-regular if every distribution solution $u$ of the equation $Pu=0$ defined in $S(0, h)$ for some $h$, is in $C^0(S(0, l))$ for some $l$, whenever $u$ belongs to $C^p(S(0, h)\frown\{x \mid (x, \xi)\leqq 0\})$, where $l(<h)$ and $p$ are independent of $u$.

In the present note we give some characterization of the $\xi$-regularity using A. Seidenberg's Theorem [1] as follows:

Theorem. *The necessary and sufficient condition for $P$ to be $\xi$-regular is the following: there are a neighbourhood $S(\xi, \delta)$, positive numbers $A$, $B$, $L$, $\alpha$ such that if for any real number $s$, for any real vector $\eta\in\Xi^n$ and for any $\xi'\in S(\xi, \delta)$*

$$A<s<B(|\eta|+1)^{\alpha} \quad \text{and} \quad |\eta|>L,$$

*then $s\xi'+i\eta$ does not satisfy the characteristic equation of $P$, i.e.,*

$$P(s\xi'+i\eta)\neq 0.$$

By Theorem and using Hörmander's considerations [2] we see the following

Corollary 1. *If $P$ is homogeneous and $Q$ is weaker than $P$ and of order $<m$, then $P+Q$ is $\xi$-regular, whenever $P$ is so.*

Corollary 2. *Let $n\geqq 3$. Then the following conditions are equivalent:*

(1) *$P+Q$ is $\xi$-regular for any $Q$ such that the order of $P>$ the order of $Q$,*

(2) *$P(\xi)\neq 0$ and if a real $\eta(\neq 0)$ satisfies the equation*

$$P(\eta)=0,$$

*then*

$$(\xi, (grad\, P))\ (\eta)\neq 0, \text{ and}$$

(3) *$P$ is of principal type and is hypo-$\xi$-regular.*

Corollary 3. *If $P$ is not hypo-elliptic, then there exists an $\xi$*

*such that P is not hypo-ξ-regular.*

2. To prove theorem we use the following Lemmas.

Lemma 1. *Let $P$ be a polynomial of $\Xi^n$ having the property: there is a continuous curve $(s(r), r)$ defined by $\{(s, r) \mid s(r)=k\cdot\log r$ and $r>L$ for some $L$ and for some $k>0\}$ such that any point $(s(r), r)$ of this curve satisfies the condition: for any $\eta\in\Xi^n$ with $|\eta|=r$, for any $\xi'\in S(\xi, \delta)$*

$$P(S(r)\xi'+i\eta)\neq 0.$$

*Then there exists positive numbers $A$, $B$, $L$ and $\alpha$ such that for any $s$ satisfying the condition:*

$$A<S<B(1+|\eta|)^{\alpha} \quad \text{and} \quad |\eta|>L$$

*and for any $\xi'\in S(\xi, \delta)$,*

$$P(s\xi'+\dot{I}\eta)\neq 0.$$

This lemma is proved applying twice Seidenberg's Theorem. (See Hörmander [2].)

Lemma 2. *If it satisfies the conclusion of Lemma 1, then $P$ is ξ-regular.*

For applying the consideration used in my paper [4], we can construct a fundamental solution $K$ of $P\left(\frac{\partial}{\partial x}\right)$ such that for some $\delta'$

$$K(x)\in C^{\infty}(\Xi^n-V(0, \delta')),$$

where $V(x, \delta')=\{y \mid (y-x, \xi')\geqq 0$ for any $\xi'\in S(\xi, \delta')\}$.

Therefore by the usual method we can conclude that $P$ is ξ-regular.

Lemma 3. *If the assumption of Lemma 1 for sufficiently small $\delta$ does not satisfied, then $P$ is not ξ-regular.*

**Proof.** We assume that $P$ is ξ-regular. Then by the closed graph theorem of Banach space and by a geometrical consideration we see that for any $\delta<\delta'$, for any positive integer $\gamma$ and for some $p$ there exists positive number $K$ such that

$$||u||_{C_0^{-\gamma}(\Lambda((h-l)\xi, \delta))}+||u||_{C^p(\Lambda(-l\xi, \delta))}\geqq K|u(0)|$$

for any solution $u$ of $Pu=0$ with $u\in C_0^{\infty}(\Lambda((h-l)\xi, \delta)$, where $\Lambda(y, \delta) =\{x \mid (x-y, \xi')\leqq 0$ and $x>-B$ for any $\xi'\in S(\xi, \delta)$ and a sufficiently large $B\}$. Now we suppose that the assumption of Lemma 1 is not satisfied. Then for any $K$ there exists a sequence $\{s_{\alpha}\xi_{\alpha}+i\eta_{\alpha}\}$ of solution $P(s_{\alpha}\xi_{\alpha}+i\eta_{\alpha})=0$ such that

$$s_{\alpha}=K\log|\eta_{\alpha}|,\ |\eta_{\alpha}|>L_{(K)} \text{ and } \xi_{\alpha}\in S(\xi, \delta),$$

where $\delta$ is sufficiently small. Now by $u(x)$ we denote the function:

$$U(x)=\sum_{\alpha} U_{\alpha}(x),$$

$$U_{\alpha}(x)=e^{(s_{\alpha}\xi_{\alpha}+i\eta_{\alpha})x+ls_{\alpha}}|\eta_{\alpha}|^{-p-1}.$$

Then we see that

($U_1$) $\sum_{\alpha} u_{\alpha}(x)$ converges absolutely in $C_0^{-\gamma}(\Lambda(h-l)\xi, \delta)$,

$(U_2)$ $\sum_\alpha u_\alpha(x)$ converges absolutely in $C^p(\Lambda(-l\xi, \delta))$ and

$(U_3)$ $\sum_\alpha u_\alpha(0)$ does not converge,

which contradicts to the above estimate.

Therefore we only have to show that $(u_1)$, $(u_2)$ and $(u_3)$ are valid. Now for $\varphi(x)\in C_0^\infty(\Lambda((h-l)\xi, \delta)$,

$$\begin{aligned}&\left|\int u_\alpha(x)\varphi(x)\,dx\right|\\ &\leqq\left|\int Q^r\Big(\frac{\partial}{\partial x}\Big)e^{(s_\alpha\xi_\alpha+i\eta_\alpha)x+ls_\alpha}(Q^r(s_\alpha\xi_\alpha+i\eta_\alpha))^{-1}|\eta_\alpha|^{-p-1}\varphi(x)\,dx\right|\\ &\leqq\left|\int e^{(s_\alpha\xi_\alpha+i\eta_\alpha)x+ls_\alpha}|Q^r(s_\alpha\xi_\alpha+i\eta_\alpha)|^{-1}|\eta_\alpha|^{-p-1}Q^r\Big(-\frac{\partial}{\partial x}\Big)\varphi(x)dx\right|\\ &\leqq L\|\varphi(x)\|_{C^r}\cdot e^{s_\alpha(\xi_\alpha\cdot x)+s_\alpha l}|\eta_\alpha|^{-r-p-1}\end{aligned}$$

for some $L$. Therefore

$$\|u_\alpha(x)\|_{C_0^{-r}(\Lambda((h-l)\xi,\,\delta))}\leqq Le^{s_\alpha((h-l)(1+\delta)+l-\frac{p+r+1}{K})}.$$

Furthermore we see that

$$\|u_\alpha(x)\|_{C^p(\Lambda(-l)\xi,\,\delta))}\leqq Le^{s_\alpha((-l)(1-\delta)+l-\frac{1}{K})}$$

and

$$|u_\alpha(0)|\geqq e^{s_\alpha(l-\frac{p+1}{K})}.$$

Hence we must choose $\{s_\alpha\xi_\alpha+i\eta_\alpha\}$ such that

$$\begin{aligned}&\frac{p+r+1}{K}+l\delta-h(1+\delta)\geqq 2\frac{\log\alpha}{s_\alpha}\\ &\frac{1}{K}-l\delta\qquad\qquad\qquad\ \geqq 2\frac{\log\alpha}{s_\alpha}\\ &\frac{p+1}{K}-l\qquad\qquad\qquad\ \leqq\frac{\log\alpha}{s_\alpha}.\end{aligned}$$

Therefore we take sufficiently large $K$ such that $\frac{p+1}{K}-l\leqq 0$ and then sufficiently small $\delta$ such that $\frac{1}{K}-l\delta\geqq\varepsilon$ for some $\varepsilon>0$ and finally sufficiently large $r$. Then if we take sufficiently large $s_\alpha$ such that $\frac{1}{2}\varepsilon s_\alpha>\log\alpha$, we see that all our requirements are satisfied.

By Lemmas 1 and 3 we see that if $P$ is $\xi$-regular, it satisfies the condition of Theorem.

3. Remark. From above it is easily seen that $P$ is $\xi$-regular if and only if for some $h$ and $l$ $(h>l>0)$ there exists a positive integer $p(h, l)$ such that for any integer $q$, any solution $u$ of $Pu=0$ is in $C^q(S(0, l))$, whenever $u\in C^0(S(0, h))$ and $u\in C^{\alpha q+p(h,l)}(S(0, l)\frown\{x\,|\,(x, \xi)\leqq 0\})$ for some $\alpha\leqq 1$. Now we shall define the general $\xi$-regularity: we say that $P$ is hypo-$\xi$-regular if every distribution solution $u$ of the equation $Pu=0$ in $C^0(S(0, h))$ for some $h$, is in

$C^{\infty}(S(0, l))$ for some $l$, whenever $u$ belongs to $C^{\infty}(S(0, h)\frown\{x \mid (x, \xi)\leqq 0\})$, where $l$ may depend upon $u$. Here we remark that from them hypo-$\xi$-regularity it does not always imply the $\xi$-regularity.

For example, we consider the operator $P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)=\frac{\partial^2}{\partial t^2}+i\frac{\partial}{\partial x}$. We easily see that $P$ is $\xi_1$-regular in 2-dim. space $\{(t, x)\}$, but not in 3-dim. space $\{(t, x, y)\}$. Now we show that $P$ is hypo-$\xi$-regular in 3-dim. space. To prove it we may suppose by a coordinate transformation that $Pu=f$, where $P\left(x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)=\frac{\partial^2}{\partial t^2}+i\frac{\partial}{\partial x}+\varepsilon x\frac{\partial}{\partial t}$, $f\in C^{\infty}_{t,x,y}(S(0, h))$, $u\in C^{2}_{t,x,y}(S(0, h))$ and $u=0$ when $t\leqq \varepsilon x^2+\varepsilon y^2$. Then by F. John's consideration [3], for some $A$

$$|||u(t, x, i\xi_3)|||_{\theta}\leq A(||(Pu)(t, x, i\xi_3)||_h)^{\alpha}(|||u(t, x, i\xi_3)|||_h)^{1-\alpha}$$

where

$$|||v(t, x)|||_{\theta}=\left\|\frac{\partial}{\partial t}v(t, x)\right\|_{\theta}+\left\|\frac{\partial}{\partial x}v(t, x)\right\|_{\theta}+||v(t, x)||_{\theta},$$

$$||v(t, x)||_{\theta}=||v(t, x)||_{L^2([0, \theta]\times R_x)},$$

$$\theta<h'<h \quad \text{and} \quad \alpha=\frac{h'-\theta}{h'}.$$

Therefore by our assumption, it implies that $||u(t, x, i\xi_3)||_{\theta}\leqq K(1+|\xi_3|)^{-k}$ for any $k$ and some $K=K(k)$, hence $u\in C^{\infty}_{y}(S(0, l))$ for some $l$, from which we see that $u\in C^{\infty}_{t,x,y}(S(0, l))$, since $P$ is $\xi$-regular in 2-dim. space $\{(t, x)\}$.

Furthermore we remark that this example satisfies the necessary and sufficient condition to be $\xi$-regular, mentioned in this section, but with $\alpha>1$.

## References


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[3] J. John: Continuous dependence on data for solutions of partial differential equations with a prescribed bound, Communications on pure and applied Math., **13** (1960).

[4] T. Shirota: On solutions of a partial differential equation with a parameter, Proc. Japan Acad., **32** (1956).