156. On the Summability Methods of Logarithmic Type

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§1. When a sequence $\{s_n\}$ is given we define the method l as follows: If

(1) $$t_0=s_0,\ t_1=s_1,$$
$$t_n=\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\quad (n\geq 2)$$

tend to a finite limit s as $n\to\infty$, we say that $\{s_n\}$ is summable (l) to s and write $\lim s_n=s(l)$. (See [3], p. 59, p. 87.)

On the other hand we define the method L as follows: If

(2) $$\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$

tends to a finite limit s as $x\to 1$ in the open interval $(0,1)$, we say that $\{s_n\}$ is summable (L) to s and write $\lim s_n=s(L)$. (See [2].)

Concerning these methods we know the following theorems.

Theorem 1. *If $\{s_n\}$ is Cesàro summable $(C,1)$ to s, then it is summable (l) to the same sum. There is a sequence summable (l) but not summable $(C,1)$.* (See [3], p. 59, [5], p. 32.]

Theorem 2. *If $\{s_n\}$ is Abel summable (A) to s, then it is summable (L) to the same sum. There is a sequence summable (L) but not summable (A).* (See [2], [3], p. 81.)

Here we establish the following theorems.

Theorem 3. *If $\{s_n\}$ is summable (l) to s, then it is summable (L) to the same sum.*

Theorem 4. *If $\{s_n\}$ is summable (l) to s, then*
$$s_n=o(n\log n).$$
Furthermore if we put
$$s_n=a_0+a_1+\cdots+a_n\quad (n\geq 0),$$
we get
$$a_n=o(n\log n)$$
from the summability (l) of $\{s_n\}$.

Theorem 5. *There is a sequence summable (L) but not summable (l).*

§2. **Proof of Theorem 3.** From (1) we get
$$t_0=s_0,\ t_1=s_1,$$
$$t_n\log n=s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\quad (n\geq 2),$$

or

$$(3)\qquad t_2\log 2=s_0+\frac{s_1}{2}+\frac{s_2}{3},$$
$$t_n\log n-t_{n-1}\log(n-1)=\frac{s_n}{n+1}\quad(n\geq 3).$$

Hence

$$(4)\qquad \frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$
$$=\frac{-1}{\log(1-x)}\left[s_0x+\frac{s_1}{2}x^2+\frac{s_2}{3}x^3+\sum_{n=3}^{\infty}\{t_n\log n-t_{n-1}\log(n-1)\}x^{n+1}\right]$$
$$=\frac{-1}{\log(1-x)}\left[t_0x+\frac{t_1}{2}x^2+\left(t_2\log 2-t_0-\frac{t_1}{2}\right)x^3-\right.$$
$$\left.-t_2x^4\log 2+\sum_{n=3}^{\infty}t_n(x^{n+1}-x^{n+2})\log n\right],$$

since, for $0<x<1$,

$$\lim_{n\to\infty}t_nx^{n+1}\log n=0$$

from the assumption of this theorem. From (4) we get

$$(5)\qquad \lim_{x\to 1-0}\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$
$$=\lim_{x\to 1-0}\frac{-x(1-x)}{\log(1-x)}\sum_{n=3}^{\infty}t_nx^n\log n.$$

Now we put

$$\psi(x)=\sum_{n=3}^{\infty}x^n\log n,$$
$$\psi_t(x)=\sum_{n=3}^{\infty}t_nx^n\log n,$$
$$\varphi(x)\begin{cases}=\dfrac{-\log(1-x)}{x(1-x)} & \text{for } 0<x<1\\ =1 & \text{for } x=0\,.\end{cases}$$

It is clear that $\psi(x)$ and $\psi_t(x)$ converge for $0\leq x<1$, since $\lim t_n=s$.

Further we have, for $0\leq x<1$,

$$\varphi(x)=1+\left(1+\frac{1}{2}\right)x+\left(1+\frac{1}{2}+\frac{1}{3}\right)x^2+\cdots\,.$$

Here we use the following

Lemma. *If* $d_n>0$, $\sum_{n=0}^{\infty}d_n=\infty$, $\sum_{n=0}^{\infty}d_nx^n$ *and* $\sum_{n=0}^{\infty}c_nx^n$ *are both convergent for* $0\leq x<1$, *and if* $\lim_{n\to\infty}\frac{c_n}{d_n}=\alpha$, $-\infty\leq\alpha\leq\infty$, *then*

$$\lim_{x\to 1-0}\frac{\sum c_nx^n}{\sum d_nx^n}=\alpha.$$

For the proof see [4], pp. 175–177.

In this lemma we put

$$d_n=1+\frac{1}{2}+\cdots+\frac{1}{n+1}\quad(n\geq 0),$$

and

$$c_0=c_1=c_2=0$$
$$c_n=t_n \log n \quad (n\geq 3).$$

Since

$$\lim_{n\to\infty}\frac{\log n}{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}=1,$$

we get

$$\lim_{x\to 1-0}\frac{\psi_t(x)}{\varphi(x)}=\lim_{n\to\infty} t_n=s,$$

from the assumption of this theorem, whence the proof is complete from (5).

Proof of Theorem 4. From (3) we get

$$s_n=(n+1)\{t_n \log n - t_{n-1}\log(n-1)\} \quad (n\geq 3).$$

Hence

$$\frac{s_n}{n\log n}=\frac{(n+1)}{n}\left\{t_n-t_{n-1}\frac{\log(n-1)}{\log n}\right\} \quad (n\geq 3),$$

which tends to 0 as $n\to\infty$ from the assumption of this theorem.

To prove the second part of this theorem we use the following formula:

$$a_n=s_n-s_{n-1}$$
$$=(n+1)t_n\log n-(2n+1)t_{n-1}\log(n-1)+nt_{n-2}\log(n-2).$$

Thus we can see similarly

$$\lim\frac{a_n}{n\log n}=0.$$

Proof of Theorem 5. We define the series $\sum_{n=0}^{\infty}a_n$ and the sequence

$$s_n=a_0+a_1+a_2+\cdots+a_n \quad (n\geq 0)$$

by the following expression

$$e^{1/(1+x)}=\sum_{n=0}^{\infty}a_n x^n.$$

This example is used to show the existence of the sequence which is summable (A) but not summable (C, r) for any r, $r>-1$. (See [3], Theorem 56.) It is known that a_n is not $O(n^r)$ for any r, whence $\{s_n\}$ is not summable (l) from Theorem 4. On the other hand $\{s_n\}$ is summable (L) from Theorem 2.

This completes the proof.

References

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