163. *Hilbert Transforms in the Stepanoff Space*

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1. **Introduction.** By the Stepanoff space we mean the set of measurable function such that for some positive number $l>0$ there exists a constant K and we have

$$(1.1)\qquad \sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f(t)|^p dt\leqq K\quad (1\leqq p\leqq\infty).$$

We denote these classes by S^p. This norm is firstly introduced by W. Stepanoff [4] for the study of almost periodic functions.

The purpose of this paper is to find under what condition does the Hilbert transform of a function of the class S^2 belong to the same class again?

The Hilbert transform is defined by the following formula

$$(1.2)\qquad \tilde{f}(x)=1/\pi\int_{-\infty}^{\infty}f(t)/(x-t)dt.$$

We understand this singular integral as the Cauchy sense. It does not always define and we assume its existence for almost all x.

One of the important property of Hilbert transform is that it commutes with translations and dilatations. These are

$$(1.3)\qquad F(t)=f(t+a)\quad\text{implies}\quad (\tilde{F})(t)=(\tilde{f})(t+a).$$

$$(1.4)\qquad F(t)=f(\lambda t)\quad\text{implies}\quad (\tilde{F})(t)=(\tilde{f})(\lambda t).$$

These properties are pointed out explicitely by M. Cotlar [1]. The author have learned this through mimeorgraphed papers presented by Dr. Y. M. Chen of the Hong-Kong University. The author thanks him for his kind considerations.

2. **Equivalence between two norms.** We consider the second norm. For all $T\geqq 1$ and all real number x, $(1/2T)\int_{-T}^{T}|f(t+x)|^p dt$ is uniformly bounded. That is, there exists a constant K such as

$$\frac{1}{2T}\int_{-T}^{T}|f(t+x)|^p dt\leqq K'\quad (-\infty<x<\infty,\ T\geqq 1).$$

And thus we get

$$(2.1)\qquad \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|f(x+t)|^p dt\leqq K'\quad \text{unif. } x.$$

We denote this uniform norm as (2.1).

Lemma 1. The two norms (1.1) and (2.1) are equivalent.

If (1.1) consists for some $l>0$ then it does also for any other $l'>0$

with other constant K''. It is enough to prove the lemma for $l=1$. It is immediate that (2.1) contains (1.1). We shall show that (1.1) leads (2.1). For any pair of $T\geqq 1$ and $-\infty<x<\infty$, we get

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}|f(x+t)|^p dt &\leqq \frac{1}{2T}\sum_{n=-[T]-1}^{[T]+1}\int_{(n-1)}^{n}|f(t+x)|^p dt\\ &\leqq \frac{1}{2T}\sum_{n=-[T]-1}^{[T]+1}\sup_{-\infty<x<\infty}\int_{x}^{x+1}|f(t)|^p dt\\ &\leqq \sup_{-\infty<x<\infty}\int_{x}^{x+1}|f(t)|^p dt\frac{1}{T}([T]+1)\\ &\leqq 2K.\end{aligned}$$

Thus we get

$$\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|f(x+t)|^p dt\leqq K \quad \text{unif. } x.$$

3. One sided Wiener's formula. Let $f(t)$ be real valued measurable function defined on $(-\infty, \infty)$. Furthermore through this section let us suppose that $f(t)$ is non-negative. This is essential for our one-sided Tauberian theorem.

We consider the two formulas:

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\int_{0}^{T} f(x+t)dt\leqq A \quad \text{unif. } x \tag{3.1}$$

and

$$\overline{\lim_{\varepsilon\to 0}}\frac{2}{\pi\varepsilon}\int_{0}^{\infty} f(x+t)\frac{\sin^2 t}{t^2}dt\leqq A' \quad \text{unif. } x. \tag{3.2}$$

We understand these as we do in preceeding section. We concern (3.1) and (3.2) with $T\geqq 1$ and $0\leqq\varepsilon\leqq 1$ respectively.

Then we have as an extenstion of the famous Winer formula:

Theorem 1. (3.1) and (3.2) are equivalent each other.

The proof can be done by running on the line of N. Wiener [5]. For the sake of completeness we repeat his arguments here. Because uniformity is essential for later arguments.

If we assume that (3.2) is satisfied. Then by the positiveness of $f(t)$, we get for any $0<\varepsilon\leqq 1$

$$\frac{2\varepsilon}{\pi}\int_{0}^{\pi/\varepsilon} f(x+t)dt\leqq 4A'/\pi^2 \quad \text{unif. } x.$$

Thus we get (3.1).

Next we assume that (3.1) is satisfied. Then we get

$$\begin{aligned}\sup_{-\infty<x<\infty}\int_{-\infty}^{\infty}\frac{f(x+t)}{1+t^2}dt &= \sup_{-\infty<x<\infty}\int_{x}^{x+1} f(t)dt\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2}\\ &\leqq \pi\sup_{-\infty<x<\infty}\int_{x}^{x+1} f(t)dt.\end{aligned} \tag{3.3}$$

And so we get

$$\frac{1}{T}\int_0^1 f(x+t)dt \leqq \frac{2}{T}\int_0^1 \frac{f(x+t)}{1+t^2}dt \leqq 2\pi K/T \to 0, \quad \text{unif. } x$$

as $T\to\infty$ and

$$\frac{2}{\pi\varepsilon}\int_0^1 f(x+t)\frac{\sin^2 \varepsilon t}{t}dt \leqq \frac{2\varepsilon}{T}\int_0^1 f(x+t)dt \leqq 4K \to 0, \quad \text{unif. } x$$

as $\varepsilon\to 0$, respectively. Thus without loss of generality we assume that $f(x+t)=0$ for $0\leqq t\leqq 1$. Then we have from (3.1) for any pair of number C and D such as $0\leqq C\leqq D\leqq\infty$

$$\int_C^D \frac{f(x+t)}{t^2}dt \leqq \frac{3A}{\max(C,1)} \quad \text{unif. } x. \tag{3.4}$$

Because by integration by parts we get

$$\begin{aligned}\int_C^D \frac{f(x+t)}{t^2}dt &= \int_{C'}^D \frac{1}{t}\, d\Big[\int_0^t f(x+\xi)\,d\xi\Big] \\ &= \frac{1}{D^2}\int_0^D f(x+\xi)d\xi - \frac{1}{C'}\int_0^{C'} f(x+\xi)d\xi + \int_{C'}^D \frac{2}{t^3}\Big[\int_0^t f(x+\xi)d\xi\Big]dt \\ &\leqq \frac{A}{C'} + A\int_{C'}^\infty \frac{2}{t^2}dt \\ &= \frac{3A}{C'}, \quad \text{unif. } x\end{aligned}$$

where we put C' instead of $\max(C,1)$.

From (3.4) we lead that the integral appeared in (3.2) exists for every x. If we devide into two parts

$$\begin{aligned}&\frac{2}{\pi\varepsilon}\int_0^\infty f(x+t)\frac{\sin^2 \varepsilon t}{t^2}dt \\ &= \frac{2}{\pi}\Big(\int_0^{\pi/\varepsilon} + \int_{\pi/\varepsilon}^\infty\Big) f(x+t)\frac{\sin^2 \varepsilon t}{t^2}dt = I_1 + I_2 \quad \text{say.}\end{aligned}$$

Then we get

$$I_1 \leqq \frac{2\varepsilon}{\pi}\int_0^{\pi/\varepsilon} f(x+t)dt \leqq 2A \quad \text{unif. } x$$

immediately and we also get from (3.4)

$$I_2 \leqq \frac{2}{\pi\varepsilon}\int_{\pi/\varepsilon}^\infty \frac{f(x+t)}{t^2}dt \leqq 6A/\pi^2 \quad \text{unif. } x.$$

Thus the proof of Theorem 1 is completed.

Furthermore if we need to prove that it is bounded from above by the same constant in (3.1) and (3.2). We can attain by the same but detailed arguments as that of N. Wiener. But we need the G. H. Hardy and J. E. Littlewood Tauberian theorem [2, pp. 180–181].

If we apply Theorem 1 to $\frac{1}{2}\{f(x+t)+f(x-t)\}$. Then we get

Corollary. Let $f(t)$ be non-negative. Then the following formulas

(3.5) $$\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T} f(x+t)dt \leqq A \quad \text{unif. } x$$

(3.6) $$\overline{\lim_{\varepsilon\to 0}}\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty} f(x+t)\frac{\sin^2 \varepsilon t}{t^2}dt \leqq A' \quad \text{unif. } x$$

are equivalent.

4. Generalized harmonic analysis in the Stepanoff Space. By W^2 we mean the class of measurable function defined on the real line such as

(4.1) $$\int_{-\infty}^{\infty}\frac{|f(t)|^2}{1+t^2}dt < \infty.$$

From (3.3) it is clear that $W^2 \supset S^2$. For this class, the generalized Fourier transform, which is introduced by N. Wiener [6], is well defined:

(4.2) $$s(u, x) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(t+x)\frac{e^{-iut}-1}{-it}dt + \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1} + \int_{1}^{A}\right] f(t+x)\frac{e^{-iut}}{-it}dt.$$

For any positive $\varepsilon > 0$ we have

(4.3) $$s(u+\varepsilon, x) - s(u-\varepsilon, x) = \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f(t+x)\frac{2\sin\varepsilon t}{t}e^{-iut}dt.$$

Applying the Plancherel theorem we have

(4.4) $$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon, x) - s(u-\varepsilon, x)|^2 du = \frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}|f(t+x)|^2\frac{\sin^2\varepsilon t}{t^2}dt.$$

Then if we apply Corollary 1 to (4.4) we get immediately

Theorem 2. Let f belong to W^2. Then the following formulas are equivalent.

(4.5) $$\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon, x) - s(u-\varepsilon, x)|^2 dx \leqq A \quad \text{unif. } x$$

and

(4.6) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x+t)|^2 dt \leqq A' \quad \text{unif. } x.$$

5. Hilbert transforms in the Stepanoff Space. For f from the class W^2, the modified Hilbert transform is well defined. That is

(5.1) $$\tilde{f}_1(t) = \frac{t+i}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{s+i}\frac{1}{t-s}ds.$$

Formally we get

(5.2) $$\tilde{f}_1(t) \sim \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}ds + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{s+i}ds = f(t) + A_f,$$

where

(5.3) $$A_f = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{s+i}ds.$$

Therefore for the existence of Hilbert transform $\tilde{f}(t)$, it is equivalent that the A_f is finitely determined. The modified one does not keep the property to commutes with translations. But if we assume that the ordinary Hilbert transform exists for a.e. x, then we get for $F(s) = f(s+a)$

$$(5.4)\qquad (\tilde{F}_1)(t) = \frac{t+i}{\pi}\int_{-\infty}^{\infty}\frac{f(s+a)}{s+i}\,\frac{ds}{t-s}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s+a)}{t-s}ds + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s+a)}{s+i}ds$$
$$= (\tilde{F})(t) + A_f(a),$$

where

$$(5.5)\qquad A_f(a) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s+a)}{s+i}ds.$$

From our preceeding papar [3, Ths. 45, 59] the $(F)(t)$ also belongs to the class W_2 and its generalized Fourier transform is well defined. We denote this by $\tilde{s}(u, a)$. Then we get

Theorem 3. Let $f(t)$ belong to the class S^2. Let us assume that its ordinary Hilbert transform exists for a.e. t. Then we have for any positive number $\varepsilon>0$,

(i) if $|u|>\varepsilon$

$$(5.6)\qquad \tilde{s}(u+\varepsilon, x) - \tilde{s}(u-\varepsilon, x) = (-i \operatorname{sign} u)\{s(u+\varepsilon, x) - s(u-\varepsilon, x)\}$$

and

(ii) if $|u|\leqq\varepsilon$

$$(5.7)\qquad \tilde{s}(u+\varepsilon, x) - \tilde{s}(u-\varepsilon, x) = i\{s(u+\varepsilon, x) - s(u-\varepsilon, x)\}$$
$$+2\,r_1(u+\varepsilon, x) + 2\,r_2(u+\varepsilon, x) - \sqrt{\frac{\pi}{2}}A_f(x),$$

where

$$(5.8)\qquad r_1(u, x) = \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{f(x+t)}{t+i}\,\frac{e^{-iut}-1}{-it}dt$$

$$(5.9)\qquad r_2(u, x) = \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{f(x+t)}{t+i}\,e^{-iut}dt.$$

Combining with Theorems 2 and 3 we attain the following result.

Theorem 4. Under the same assumptions as Theorem 3, the necessary and sufficient for $\tilde{f}(t)$ to belong to the same class S^2 is

$$(5.10)\qquad \overline{\lim_{\varepsilon\to 0}}\frac{1}{2\varepsilon}\int_0^{2\varepsilon}\left|\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{f(t+x)}{t+i}e^{-iut}dt - \sqrt{\frac{\pi}{2}}A_f(x)\right|^2 du \leqq K, \text{ unif. } x.$$

For the proof of Theorem 4 it is enough to show that

$$(5.11)\qquad \overline{\lim_{\varepsilon\to 0}}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_1(u+\varepsilon, x)|^2du \leqq K \quad \text{unif. } x.$$

We get

$$\int_{-\varepsilon}^{\varepsilon}|r_1(u+\varepsilon, x)|^2du = \int_0^{2\varepsilon}\left|\int_{-\infty}^{\infty}\frac{f(x+t)}{t+i}\,\frac{e^{-iut}-1}{-it}dt\right|^2du$$

$$\leqq \int_0^{2\varepsilon}\left(\int_{-\infty}^{\infty}\frac{|f(x+t)|^2}{1+t^2}dt\right)\left(\int_{-\infty}^{\infty}\left|\frac{e^{-iut}-1}{-it}\right|^2 dt\right)du$$

$$\leqq \sup_{-\infty<x<\infty}\int_{-\infty}^{\infty}\frac{|f(x+t)|^2}{1+t^2}dt\cdot O\left(\int_0^{2\varepsilon}u\,du\right)\leqq \sup_{-\infty<x<\infty}\int_x^{x+1}|f(t)|^2dt\cdot O(\varepsilon^2).$$

Thus we get

$$(5.11)' \qquad \overline{\lim_{\varepsilon\to 0}}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_1(u+\varepsilon, x)|^2\,du=o(1), \quad \text{unif. } x.$$

By the similar arguments we obtain

Theorem 5. Let $f(t)$ be S^2 almost periodic and have no spectre at $u=0$. Let us assume that the Hilbert transform $\tilde{f}(t)$ exists a.e. t. Then the necessary and sufficient condition for $\tilde{f}(t)$ to be also S^2 almost periodic and to have no spectre at $u=0$, is

$$(5.12)\ \overline{\lim_{\varepsilon\to 0}}\frac{1}{2\varepsilon}\int_0^{2\varepsilon}\left|\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{f(t+x)}{t+i}e^{-iut}dt-\sqrt{\frac{\pi}{2}}A_f(x)\right|^2 du=o(1)\ \text{unif.}$$

Let the associated Fourier series of $f(t)$ be

$$(5.13) \qquad f(t)\sim{\sum}' a_n e^{i\lambda_n t}.$$

Then we get

$$(5.14) \qquad \tilde{f}(t)\sim{\sum}'(-i\,\text{sign}\,\lambda_n)\ a_n e^{i\lambda_n t}.$$

The prime mean that the term $n=0$ is excluded from the summation.

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