105. On the Spectra of Some Non-linear Operators

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Let R be a real Banach space, K be a completely continuous, linear operator defined on R into itself and $\mathfrak{f}$ be a (in general, non-linear) continuous operator defined on R into itself.

In this note, we will study the properties of proper values and proper vectors of the operator $H=K\mathfrak{f}$. The integral operators of Hammerstein type are of this type.[1]

We denote by $S(H)$ and $S(K)$ the set of proper values of H and K respectively. We denote the set of proper vectors belonging to $\lambda\in S(H)$ or $\lambda\in S(K)$ by $E_\lambda(H)$ or $E_\lambda(K)$ respectively. We know that $S(K)$ is bounded, discrete and $E_\lambda(K)$ is finite-dimensional.

The purpose of this paper is to study in what cases $S(H)$ is bounded or discrete or, in the case of Hilbert spaces, $E_\lambda(H)$ contains finite number of orthogonal elements.

For this purpose, we see that the case when $H0\neq 0$ is exceptional, because we have the following

Theorem 1. *Let $H=K\mathfrak{f}$ be defined on a Banach space and $H0\neq 0$. If there exist numbers $a>0$ and $b>0$ such that*

$$(\sharp) \qquad \|\mathfrak{f}\phi\|\leqq a+b\|\phi\|$$

for every $\phi\in R$, then $|\lambda|\geqq(a+b)\|K\|$ implies $\lambda\in S(H)$.

The proof is omitted, because this is an easy consequence of Schauder's fixed point theorem. In case of integral operators of Hammerstein type, defined on $L_p(p>1)$ or Orlicz spaces, the condition ($\sharp$) is equivalent to the fact $\mathfrak{f}$ is defined on the whole space. For this, we refer [1, 2].

In the sequel, we assume that $\mathfrak{f}0=0$.

§1. *Boundedness of* $S(H)$.

Theorem 2. *Let $H=K\mathfrak{f}$ be defined on a Banach space R. If the operator $\mathfrak{f}$ with ($\sharp$) be Fréchet-differentiable at* 0 *and the gradient $\nabla\mathfrak{f}0$ be continuous, then $S(H)$ is bounded.*

Proof. Since $\mathfrak{f}$ is Fréchet-differentiable, for any positive number $\varepsilon>0$ there exists $\delta>0$ such that

$$\|\mathfrak{f}\phi-(\nabla\mathfrak{f}0)\phi\|\leqq\varepsilon\|\phi\| \qquad \text{if } \|\phi\|<\delta.$$

Therefore, $\|\phi\|<\delta$ implies that

1) For $K\phi(s)=\int K(s,t)\phi(t)\,dt$ and $\mathfrak{f}\phi(t)=f(t,\phi(t))$, the integral operator $H\phi(s)=K\mathfrak{f}\phi(s)=\int K(s,t)\mathfrak{f}(t,\phi(t))\,dt$ is of Hammerstein type. In the remarks of this paper, we consider operators of this type.

$$(*) \qquad \| \mathfrak{f}\phi \| \leqq (\| \nabla \mathfrak{f} 0 \| + \varepsilon) \| \phi \|,$$

because the operator $\nabla \mathfrak{f}0$ is continuous. If $S(H)$ is not bounded, then there exists $\lambda \in S(H)$ such that $\phi \in E_\lambda(H)$ implies $\| \phi \| < \delta$, because, by the condition (♯), we have

$$(|\lambda| - b \| K \|) \| \phi \| \leqq a \| K \|.$$

Therefore, for such λ, we have by (*) that $\phi \in E_\lambda(H)$ implies $|\lambda| \cdot \| \phi \| \leqq \| K \| \cdot \| \mathfrak{f}\phi \| \leqq \| K \| (\| \nabla \mathfrak{f}0 \| + \varepsilon) \| \phi \|$, namely, $|\lambda| \leqq \| K \| (\| \nabla \mathfrak{f}0 \| + \varepsilon)$, which means that $S(H)$ is bounded.

Remark 1. In $L_2[0,1]$, consider $H = K\mathfrak{f}$ with $K(s,t) = st$ and $\mathfrak{f}\phi(t) = \mathfrak{f}(\phi(t))$ where $f(u) = \sqrt{|u|}$. Then the operator $\mathfrak{f}$ satisfies (♯) but is not Fréchet-differentiable at 0. It is easy to see that $S(H)$ contains every positive numbers, namely, $S(H)$ is not bounded.

Remark 2. In $L_2[0,1]$, consider $H = K\mathfrak{f}$ with $K(s,t) = st$ and $f(u) = u + u^2$. Then the operator $\mathfrak{f}$ is Fréchet-differentiable at 0 but does not satisfy (♯). It is easy to see that real numbers except for 0 and 1/3 are in $S(H)$, namely, $S(H)$ is not bounded.

§2. *Finiteness of numbers of orthogonal elements in* $E_\lambda(H)$. Since H is non-linear, the set $E_\lambda(H)$ is, in general, not linear. Therefore, instead of considering the dimension of $E_\lambda(H)$, we will find the conditions under which, in the case of Hilbert space, $E_\lambda(H)$ contains finite number of mutually orthogonal elements.

Theorem 3. *Let $H = K\mathfrak{f}$ be defined on a Hilbert space R and $\lambda \in S(H)$. Then, for any numbers $a > 0$ and $b > 0$ with $a > b$, the number of orthogonal elements in $E_\lambda(H)$ such that $0 < a \leqq \| \phi \| \leqq b$ is at most finite.*

Proof. For positive numbers a and b with $a < b$, put

$$E_\lambda^{(a,b)} = \{\phi \in E_\lambda(H) : a \leqq \| \phi \| \leqq b\}.$$

Since the operator H is compact, the set $HE_\lambda^{(a,b)}$ is compact. If $E_\lambda^{(a,b)}$ contains infinite number of orthogonal elements, then there exists an orthogonal sequence ϕ_n in $E_\lambda^{(a,b)}$ such that $\lim_{n\to\infty} \phi_n = \phi$ where ϕ is in $E_\lambda^{(a,b)}$. Evidently, ϕ must be zero. This contradicts the assumption that $0 < a \leqq \| \phi \|$.

Next, we will estimate the number of such orthogonal elements in $E_\lambda(H)$ that are bounded above. For this purpose, we, at first, prove the following

Lemma 1. *Let $H = K\mathfrak{f}$ be defined on a Banach space R. We assume that $\mathfrak{f}$ be Fréchet-differentiable at* 0 *and the gradient $\nabla \mathfrak{f}0$ be bounded. Then, $\inf_{\phi \in E_\lambda(H)} \| \phi \| = 0$ for some $\lambda \in S(H)$, $\lambda \neq 0$, implies $\lambda \in S(\nabla H 0)$.*

Proof. Since, $\inf_{\phi \in E_\lambda(H)} \| \phi \| = 0$, we can find $\phi_n \in E_\lambda(H)$ $(n = 1, 2, \cdots)$ such that $\lim_{n\to\infty} \phi_n = 0$. As $\mathfrak{f}0 = 0$, it follows from the definition of $\nabla \mathfrak{f}0$ that

$$\mathfrak{f}\phi_n = \mathfrak{f}\phi_n - \mathfrak{f}0 = (\nabla \mathfrak{f}0)\phi_n + \mathrm{r}(\phi_n)$$

where $\lim_{n\to\infty} \mathrm{r}(\phi_n)/\|\phi_n\| = 0$. Namely, we have

$$\lim_{n\to\infty} \|\mathfrak{f}\phi_n - (\nabla \mathfrak{f}0)\phi_n\| / \|\phi_n\| = 0.$$

As K is linear and continuous, it follows that

$$\lim_{n\to\infty} \| H\phi_n - (\nabla H0)\phi_n \| / \|\phi_n\| = 0.$$

Moreover, since $H\phi_n = \lambda\phi_n$, we have

$$\lim_{n\to\infty} \left\| \lambda \frac{\phi_n}{\|\phi_n\|} - (\nabla H0)\frac{\phi_n}{\|\phi_n\|} \right\| = 0.$$

On the other hand, since $\nabla\mathfrak{f}0$ is a bounded operator, $\nabla H0$ is compact, and hence it follows that there exists a subsequence ϕ_{n_i} such that

$$\lim_{i\to\infty} (\nabla H0)\phi_{n_i} / \|\phi_{n_i}\| = \phi_0$$

for some $\phi_0 \in R$. This ϕ_0 belongs to $E_\lambda(\nabla H0)$, because, since

$$\lim_{i\to\infty} \lambda\phi_{n_i} / \|\phi_{n_i}\| = \phi_0, \qquad \|\phi_0\| = |\lambda|,$$

and we have

$$(\nabla H0)\phi_0 = \lim_{i\to\infty} (\nabla H0)(\lambda\phi_{n_i} / \|\phi_{n_i}\|) = \lambda\phi_0.$$

Therefore, $\lambda \in S(\nabla H0)$.

Remark. The inverse of this lemma is not true. For example, in $L_2\left[0, \frac{\pi}{2}\right]$, consider $H = K\mathfrak{f}$ where $K(s, t) = st$ and $f(u) = \sin u$. Then, K is completely continuous and $\mathfrak{f}$ is bounded. It is easy to see that $\frac{1}{3} \in S(\nabla H0)$ and $\inf_{\phi\in E_{\frac{1}{3}}} \|\phi\| > 0$.

Theorem 4. *Let $H = K\mathfrak{f}$ be defined on a Hilbert space R. We assume that $\mathfrak{f}$ be Fréchet-differentiable at* 0 *and the operator $\nabla\mathfrak{f}0$ be bounded. Then, if $\lambda \overline{\in} S(\nabla H0)$, for any positive number a, the number of orthogonal elements in $E_\lambda(H)$ such that $\|\phi\| \leqq a$ is at most finite.*

Proof. If the set $E_\lambda^a = \{\phi \in E_\lambda(H) : \|\phi\| \leqq a\}$ contains infinite number of mutually orthogonal elements, then there exists an orthogonal sequence $\phi_n \in E_\lambda^a$ such that $\lim_{n\to\infty} \phi_n = 0$, since E_λ^a is compact. This contradicts the fact that $\inf_{\phi\in E_\lambda(H)} \|\phi\| > 0$.

Remark. If the operator $\mathfrak{f}$ satisfies the condition (♯), we have $\|\phi\| \leqq a\|K\| / (|\lambda| - b\|K\|)$ for any $\phi \in E_\lambda(H)$ and every λ such that $|\lambda| > b\|K\|$. On the other hand, the proper values of the linear, completely continuous operator $\nabla H0$ is bounded above by $\|\nabla H0\|$. Therefore, for such λ that $|\lambda| > \max.\{b\|K\|, \|\nabla H0\|\}$, the set $E_\lambda(H)$ is a bounded set and $\lambda \overline{\in} S(\nabla H0)$, which shows that $E_\lambda(H)$ contains at most finite number of mutually orthogonal elements.

This theorem shows the close connection between $S(\nabla H0)$ and $E_\lambda(H)$. The next theorem will help us to make more direct consideration.

Theorem 5. *Let $H=K\mathfrak{f}$ be defined on a Banach space R. We assume that the operator $\mathfrak{f}$ be Fréchet-differentiable at each point $\phi\in R$ and the gradient mappings $\nabla\mathfrak{f}\phi$ be continuous. If $E_\lambda(H)$ is bounded and contains infinite number of elements, then $\lambda\in S(\nabla H\phi)$ for some $\phi\in E_\lambda(H)$.*

Proof. Since $E_\lambda(H)$ is bounded and $\frac{1}{\lambda}H(E_\lambda(H))=E_\lambda(H)$, $E_\lambda(H)$ is compact, because H is a compact operator. Therefore, we can select a sequence $\phi_n\in E_\lambda(H)$ such that $\lim_{n\to\infty}\phi_n=\phi_0$ for some $\phi_0\in R$. Since $E_\lambda(H)$ is closed, $\phi_0\in E_\lambda(H)$. We can prove that $\lambda\in S(\nabla H\phi_0)$ by the same method used in the proof of Lemma 1.

§3. *Discreteness of $S(H)$.* In the case of $S(K)$, it contains no intervals. In this section, we want to characterize the class of H whose $S(H)$ contains no intervals. The next lemma is suggestive.

Lemma 2. *Let $H=K\mathfrak{f}$ be defined on a Hilbert space R and $K(R)$ be one-dimensional, namely, $K(R)=\{a\phi_0: \|\phi_0\|=1, -\infty<a<+\infty\}$. We assume that the function $(H(a\phi_0), \phi_0)/a$ is continuous as a function of a. Then, if $S(H)$ is not empty and contains no intervals, the operator HK is linear.*

Proof. If $(H(a\phi_0), \phi_0)=\lambda a$, $a\neq 0$, then, since $H(a\phi_0)\in K(R)$, $H(a\phi_0)=b\phi_0$, and hence it follows that

$$\lambda a=(H(a\phi_0), \phi_0)=(b\phi_0, \phi_0)=b.$$

Namely, $(H(a\phi_0), \phi_0)=\lambda a$, $a\neq 0$, is equivalent to $\lambda\in S(H)$. Therefore, if the range of the function $(H(a\phi_0), \phi_0)/a$ contains two different numbers, then $S(H)$ contains at least one interval. Since $S(H)$ contains no intervals, the function $(H(a\phi_0), \phi_0)/a$ is constant, namely,

$$(H(a\phi_0), \phi_0)=\lambda a \qquad (-\infty<a<+\infty),$$

which shows that $H(a\phi_0)=\lambda a\phi_0$ $(-\infty<a<+\infty)$. For any $\phi\in R$, take such a number a that $K\phi=a\phi_0$. Then,

$$HK\phi=H(a\phi_0)=\lambda a\phi_0=\lambda K\phi,$$

namely, $HK=\lambda K$, which shows that HK is linear.

The case when $K(R)$ is multi-dimensional is more complicated. We leave the detailed study to a late paper.[2] Here, we write a direct consequence of Lemma 2.

Theorem 6. *Let $H=K\mathfrak{f}$ be defined on a Hilbert space R, ψ_n $(n=1, 2,\cdots)$ be the proper functions of K and μ_n $(n=1, 2,\cdots)$ be the proper values of K to which the ψ_n belongs. We assume that the functions $(H(a\psi_n), \psi_n)/a$ are continuous with respect to a. If $S(H)$ contains no intervals and the operator $\mathfrak{f}$ is invariant on each $E_{\mu_n}(K)$, then H is linear on each $E_{\mu_n}(K)$.*

2) On the spectra of some Non-linear operators, III, to appear on the Yokohama Mathematical Journal.

Proof. Since $\mathfrak{f}$ is invariant on $E_{\mu_n}(K)$, the fact that $H(a\psi_n) = \lambda a\psi_n$ for some $a \neq 0$ is equivalent to $(H(a\psi_n), \psi_n)/a = \lambda$. Since each of these functions is continuous with respect to a, the fact that $S(H)$ contains no intervals implies that each of these functions is constant. Namely, $H(a\psi_n) = \lambda_n a\psi_n$ ($n=1, 2, \cdots; -\infty < a < +\infty$), which means that H is linear on $E_{\mu_n}(K)$.

References

[1] M. A. Krasnoseliski: Topological Method in the Theory of Non-linear Intergral Equations, Moscow (1956).

[2] M. M. Vainberg and I. V. Shragin: Nemyckii Operator and its Potential in Orlicz Spaces, Doklady Acad. Nauk (n.s.), **120**, no. 5 (1958).