70. Remarks on Knots with Two Bridges

By Kunio MURASUGI
Hosei University, Tokyo

(Comm. by K. KUNUGI, M.J.A., June 12, 1961)

§1. Introduction. In 1954, H. Schubert introduced the new numerical knot invariant, called the bridge number of the knot [6]. Then he completely classified the knots with two bridges [7]. He assigned two integers α and β to a knot k with two bridges. α is called a *torsion*, which is the same as the well-known second torsion number of k, and the other β is called "Kreuzungsklasse", whose new interpretation will be given in §2 in this note. As indicated by Schubert, the pair (α, β) will be called the *normal form* of k, where $\alpha>|\beta|>0$. After §3 the non-cyclic covering space $\mathfrak{F}$ unbranched along k will be considered following Bankwitz and Schumann [1]. Their discussion indicating that $\mathfrak{F}$ characterizes the knot plays an important role in classifying two knots of the same Alexander polynomial, as has been shown in their paper [1]. In §4 it will be shown that the Alexander polynomial over the Betti group of $\mathfrak{F}$ can be found based on the results in §3 following [3, III].

§2. Group presentation. Let k be a knot with two bridges of the normal form (α, β) and let K be its bridged projection. Let G be the knot group of k. The presentation of G will now be given based on K. K has $4p$ double points in which $2p$ double points lie in AB and the others lie in CD, where AB, CD are the *bridges*. These $4p$ double points will be named $X_1, X_2, \cdots, X_{2p}$ on AB, and $Y_1, Y_2, \cdots, Y_{2p}$ on CD in order of the direction of K starting at A. Then the over-presentation of G will be given by $G=(a, b\colon R, S)$, where $R=LaL^{-1}b^{-1}$, $S=MbM^{-1}a^{-1}$, $L=a^{\varepsilon_1}b^{\eta_1}a^{\varepsilon_2}b^{\eta_2}\cdots a^{\varepsilon_p}b^{\eta_p}$, $\varepsilon_i, \eta_j=+1$ or -1 for all i, j, and M is an element of G of the same type as L (cf. [4]), i.e. G is a group generated by two generators a, b and has two defining relations $R=1, S=1$. Since one of R, S is implied by the other, G can be considered as the group of a single defining relation R. And $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$ are defined as 1 or -1 depending on whether AB overpasses at $Y_1, Y_2, \cdots, Y_p$ from left to right or from right to left, and $\eta_1, \eta_2, \cdots, \eta_p$ are defined similarly. Thus it follows that

(2.1) G has a presentation as follows:

$$G=(a, b\colon R),\text{ where } R=LaL^{-1}b^{-1}.$$

In connection with this presentation, it follows that

(2.2) $\qquad 2p+1$ equals α.

Since ε_i, η_j are either $+1$ or -1, the series of these exponents $S=\{\varepsilon_1, \eta_1, \varepsilon_2, \eta_2, \cdots, \varepsilon_p, \eta_p\}$ can be considered as the *series of signs*. Then from the construction of the bridged projection, it follows easily that

(2.3) S is symmetric, i.e. $\varepsilon_1=\eta_p, \eta_1=\varepsilon_p, \cdots, \varepsilon_r=\eta_{p+1-r}, \cdots$.

Moreover, we obtain the following result.

(2.4) The number $\sigma(S)$ of the changes of sign in S equals $|\beta|-1$. $\sigma(S)$ is defined as

$$\sigma(S)=\frac{1}{2}\left\{\sum_{i=1}^{p}|\varepsilon_i-\eta_i|+\sum_{j=1}^{p-1}|\eta_j-\varepsilon_{j+1}|\right\}.$$

§ 3. Covering space. Consider a subgroup F of index $\nu(<\infty)$ in a group G and let its right coset be F_i, where $F_1=F$. In each coset F_i select a representative element $|F_i|$, with $|F_1|=1$. Then it is well known that F determines a representation ρ of G upon a transitive group of permutations of the symbols $1, 2, \cdots, \nu$. Conversely, given any representation ρ of G, we can find the subgroup F corresponding to ρ. In the case where G is a knot group of a knot k, any subgroup determines a covering space unbranched along k. Let k be a knot with two bridges. Then its knot group has a presentation as is shown in (2.1).

Now consider a representation ρ of G as follows:

(3.1)
$$a^\rho=(2\ 2p+1)(3\ 2p)\cdots(p+1\ p+2),$$
$$b^\rho=(1\ 2p+1)(2\ 2p)\cdots(p\ p+2).$$

ρ determines the subgroup F of index $2p+1$ of G. Since elements $a, b^2, (ba^{-1})^{2p+1}$ are contained in F, we can select the coset representative elements as follows:

(3.2)
$$|F_1|=1,$$
$$|F_i|=(ba^{-1})^{i-1}, \text{ for } i=2, \cdots, p+1,$$
$$|F_j|=(ba^{-1})^{2p-j+1}b, \text{ for } j=p+2, \cdots, 2p,$$
$$|F_{2p+1}|=b.$$

Thus F has the following presentation:

$4p+2$ generators: $a_i=|F_i|\,a\,|F_ia|^{-1}$ for $i=1, \cdots, 2p+1$.
$b_i=|F_i|\,b\,|F_ib|^{-1}$,

$2p+1$ defining relations

$$R_i=|F_i|\,R\,|F_i|^{-1}, \quad \text{for } i=1, \cdots, 2p+1.$$

In this presentation, it should be noted that $2p$ generators $a_2, \cdots, a_{p+1}, b_1, \cdots, b_p$ are trivial. Hence F has $2p+2$ non-trivial generators and $2p+1$ defining relations.

Now consider the 1st homology group $F/[F,F]$ of F, $[F,F]$ denoting the commutator subgroup. To determine the structure of $F/[F,F]$, a homomorphism ω will be introduced [3, III]

Let X be the free group generated by two generators, a, b, and let X^* be the free group generated by $4p+2$ generators $a_1, a_2, \cdots,$

$a_{2p+1}, b_1, \cdots, b_{2p+1}$. Let $\mathfrak{M}(X^*)$ be denoted by a ring of $(2p+1)\times(2p+1)$ matrices over the (integral) group ring JX^*. Then a homomorphism

$$\omega: JX \to \mathfrak{M}(X^*)$$

is defined as

$$(3.3) \qquad \begin{aligned} a^\omega &= \| \delta_{ij}(a)a_i \|_{i,j=1,2,\cdots,2p+1}, \\ b^\omega &= \| \delta_{ij}(b)b_i \|_{i,j=1,2,\cdots,2p+1}, \end{aligned}$$

where $\delta_{ij}(x)$ is defined as 1 or 0 depending on whether $F_i x = $ or $\neq F_j$.

Then it follows [3, III]:

(3.4) The torsion numbers of $F/[F,F]$ are the invariant factors of $\left\| \frac{\partial R}{\partial a} \frac{\partial R}{\partial b} \right\|^{\omega_0}$ [1] and the Betti number of $F/[F,F]$ is equal to the nullity of $\left\| \frac{\partial R}{\partial a} \frac{\partial R}{\partial b} \right\|^{\omega_0}$ decreased by $2p$, where $\omega_0 = o\omega$, o being a homomorphism from $\mathfrak{M}(X^*)$ into $\mathfrak{M}$ (1).

From (3.4), the following Lemma will be shown.

[Lemma 3.1] The Betti number of $F/[F,F]$ is equal to $p+1$ and the torsion numbers are all trivial.

§ 4. Alexander polynomials. In § 3, it has been known that $F/[F,F]=H$ is the free abelian group generated by $p+1$ generators. Hence we can find the Jacobian matrix at ψ, the abelianizing homomorphism from JF into JH[3, II]. It is immediately known [1] that if the generators of H are denoted by $t_1, t_2, \cdots, t_{p+1}$, then $\psi(a_1)=\psi(b_{p+1}) = t_1$, $\psi(a_{p+2})=\psi(b_{2p+1})=t_2$, $\psi(a_{p+3})=\psi(b_{2p})=t_3, \cdots$, $\psi(a_{2p+1})=\psi(b_{p+2}) = t_{p+1}$. First of all, let F^* be the free product of two groups F and T, where T is a *free group* generated by the trivial generators $a_2, a_2, \cdots, a_{p+1}, b_1, \cdots, b_p$. Then the Jacobian matrix of F^* is given by $\left\| \frac{\partial R}{\partial a} \frac{\partial R}{\partial b} \right\|^{\phi^*\omega}$, where ϕ^* is a homomorphism from $\mathfrak{M}(X^*)$ into $\mathfrak{M}(F^*)$. Then this matrix is equivalent to $\| O M_F \|$, where O denotes the null matrix of $2p+1$ rows and p columns, and M_F is a required Jacobian of F [3, III]. In particular, introducing the homomorphism

$$\sigma_0: JH \to JZ,$$

where Z is an infinite cyclic group generated by t, defined as $\sigma_0(t_i) = t$ for all i, we have a Jacobian of F at $\sigma_0\Psi$.

Specially we can easily show that

$$(4.1) \qquad L^{\tilde{\omega}} (L^t)^{\omega_0} \overline{\left(\frac{\partial R}{\partial b}\right)^{\tilde{\omega}}} + \left(\frac{\partial R}{\partial a}\right)^{\tilde{\omega}} (L^t)^{\omega_0} = 0,$$

where L^t denotes the transposed matrix of L, the bar over the symbol means *conjugation*,[2] and $\tilde{\omega}$ denotes the homomorphism from

1) ∂ denotes the *free differential* introduced in [3, I].

2) See [2].

F^* into $\mathfrak{M}(Z)$.

Thus the $\tilde{\nabla}$-polynomial[3] $\tilde{\nabla}(t)$ can be found from $\left(\frac{\partial R}{\partial b}\right)^{\tilde{\omega}}$ and $\tilde{\nabla}(t)$ will characterize the original knot.

Example 1. As is well known, two knots 7_4 and 9_2 have the same Alexander polynomials $4-7t+4t^2$ [5]. However, their $\tilde{\nabla}$-polynomials are

$$7_4\colon\ \tilde{\nabla}(t)=4(1+t)^2(5-6t+5t^2)$$
$$9_2\colon\ \tilde{\nabla}(t)=16(2-3t+6t^2-3t^3+2t^4).$$

Hence these have different $\tilde{\nabla}$-polynomials, which suggests that the $\tilde{\nabla}$-polynomial is not obtainable from the original Alexander polynomial in any simple way.

Example 2. The $\tilde{\nabla}$-polynomial of a torus knot of type $(2m+1, 2)$ is

$$\tilde{\nabla}(t)=(1+t^{m+1})^m(1+t+t^2+t^3+\cdots+t^m)^{m-1}.$$

This coincides with the ∇-polynomial of the closed braid in S^3 which is constructed from $p+1$ strings by twisting $2(p+1)$ times (cf. [1]).

References

[1] C. Bankwitz and H. G. Schumann: Über Viergeflechte, Abh. Hamb., **10**, 263–284 (1934).

[2] F. Hosokawa: On ∇-polynomials of links, Osaka Math. J., **10**, 273–282 (1958).

[3] R. H. Fox: Free differential calculus, I, Ann. of Math., **57**, 547–560 (1953); II, ibid., **59**, 196–210 (1954); III, ibid., **64**, 407–419 (1956).

[4] R. H. Fox and G. Torres: Dual representation of the group of a knot, Ann. of Math., **59**, 211–218 (1954).

[5] K. Reidemeister: Knotentheorie, Chelsea (1948).

[6] H. Schubert: Über eine numerische Knoteninvariante, Math. Zeit., **61**, 245–288 (1954).

[7] ——: Knoten mit zwei Brücken, Math. Zeit., **65**, 133–170 (1956).

3) Letting the Alexander polynomial over the Betti group of F be denoted by $\tilde{\Delta}(t_1, t_2, \cdots, t_{p+1})$, $\tilde{\nabla}(t)$ is defined as $\tilde{\nabla}(t)=\tilde{\Delta}(t, t, \cdots, t)$ for $p<2$, and $\tilde{\nabla}(t)=\tilde{\Delta}(t, \cdots, t)/(1-t)^{p-1}$ for $p\geqq 2$. Cf. [2].