46. *Finite-to-one Closed Mappings and Dimension. IV*

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We have published in the Proceedings three notes under this title with sketches of proofs or no proofs. The detail of the content stated there will be published, under the title 'Mappings of finite order and dimension theory', in forthcoming Japanese Journal of Mathematics. We have mainly been concerned with finite-to-one closed mappings between metric spaces. We shall notice in this note a deep relation between finite-to-one closed mappings defined on non-metrizable spaces and the inductive dimension. The detail of the content of the present note will be published in another place. In the following n denotes a non-negative integer.

Let R be a topological space. We define inductively the small and the large inductive dimension of R, $\operatorname{ind} R$ and $\operatorname{Ind} R$, as follows. For the empty set ϕ let $\operatorname{ind}\phi=\operatorname{Ind}\phi=-1$. We call $\operatorname{ind} R\leq n$, if for any point x and an open set G with $x\in G$ there exists an open set H with $x\in H\subset G$ such that $\operatorname{ind}(\overline{H}-H)\leq n-1$. We call $\operatorname{Ind} R\leq n$, if for any pair $F\subset G$ of a closed set F and an open set G there exists an open set H with $F\subset H\subset G$ such that $\operatorname{Ind}(\overline{H}-H)\leq n-1$.

Let $\mathfrak{F}=\{F_\alpha;\ \alpha\in A\}$ be a collection of subsets of R. Then the order of $\mathfrak{F}$ at x, $\operatorname{order}(x,\mathfrak{F})$, is the number of elements of $\mathfrak{F}$ which contain x. The order of $\mathfrak{F}$, $\operatorname{order}\mathfrak{F}$, is the supremum of $\{\operatorname{order}(x,\mathfrak{F});\ x\in R\}$. The star of H, a subset of R, with respect to $\mathfrak{F}$, $\mathrm{S}(H,\mathfrak{F})$, is the sum of $F\in\mathfrak{F}$ with $H\frown F\neq\phi$. Let S be a subset of R. Then the restriction of $\mathfrak{F}$ to S, $\mathfrak{F}\wedge S$, is the collection $\{F_\alpha\frown S;\ \alpha\in A\}$. Let $\mathcal{F}=\{\mathfrak{F}_\lambda;\ \lambda\in\Lambda\}$ be a system of collections of subsets of R. Then the order of $\mathcal{F}$, $\operatorname{order}\mathcal{F}$, is the supremum of $\{\operatorname{order}\mathfrak{F}_\lambda;\ \lambda\in\Lambda\}$.

Definition 1. Let $\mathcal{F}=\{\mathfrak{F}_\lambda;\ \lambda\in\Lambda\}$ be a system of coverings of a topological space R. $\mathcal{F}$ is called to *follow out the topology of R locally, globally* and *fully*, if the following conditions are respectively satisfied.
(1) For any point x of R and any open set G with $x\in G$ there exists $\lambda\in\Lambda$ with $\mathrm{S}(x,\mathfrak{F}_\lambda)\subset G$.
(2) For any pair $F\subset G$ of a closed set F and an open set G of R there exists $\lambda\in\Lambda$ with $\mathrm{S}(F,\mathfrak{F}_\lambda)\subset G$.
(3) For any open covering $\mathfrak{G}$ of R there exists $\lambda\in\Lambda$ such that $\mathfrak{F}_\lambda$ refines $\mathfrak{G}$.

Definition 2. Let $\mathcal{F}=\{\mathfrak{F}_\lambda=\{F_\alpha;\ \alpha\in A_\lambda\};\ \lambda\in\Lambda\}$ be a system of coverings of a topological space R. $\mathcal{F}$ is called a directed family (with

$\{A_\lambda, f_{\lambda\mu}\}$) if the following conditions are satisfied.

(4) Λ is a directed set.

(5) For any pair $\lambda < \mu$ there exists $f_{\mu\lambda}: A_\mu \to A_\lambda$ such that $\{A_\lambda, f_{\mu\lambda}; \lambda \in \Lambda\}$ forms an inverse limiting system of discrete spaces A_λ.

(6) For any ordered pair $\lambda < \mu$ and any $\alpha \in A_\lambda$ it holds that $F_\alpha = \cup\{F_\beta; f_{\mu\lambda}(\beta) = \alpha\}$.

Theorem 1. *If a topological space R has a directed family of locally finite closed coverings of order $\leq n+1$ which follows out the topology of R locally, then R is a regular space with* $\operatorname{ind} R \leq n$.

Theorem 2. *If a topological space R has a directed family of locally finite closed coverings of order $\leq n+1$ which follows out the topology of R globally, then R is a normal space with* $\operatorname{Ind} R \leq n$.

Theorem 3. *If a topological space R has a directed family of locally finite closed coverings of order $\leq n+1$ which follows out the topology of R fully, then R is a paracompact Hausdorff space with* $\operatorname{Ind} R \leq n$.

Theorem 4. *If a directed family $\mathcal{F} = \{\mathfrak{F}_\lambda = \{F_\alpha; \alpha \in A_\lambda\}; \lambda \in \Lambda\}$ of locally finite closed coverings of a topological space R with* order $\mathcal{F} \leq n+1$ *which follows out the topology of R locally and globally, then for any $\lambda \in \Lambda$ and any mutually different indices $\alpha_1, \cdots, \alpha_m$ of A_λ, $1 \leq m \leq n+1$, we get* $\operatorname{ind} \bigcap_{i=1}^m F_{\alpha_i} \leq n-m+1$ *and* $\operatorname{Ind} \bigcap_{i=1}^m F_{\alpha_i} \leq n-m+1$ *respectively.*

Theorem 5. *If a non-empty topological space R has a directed family of locally finite closed coverings of order $\leq n+1$ which follows out the topology of R locally, then there exist a completely regular space A with* $\operatorname{ind} A = 0$ *and a closed continuous mapping f of A onto R with* order $f \leq n+1$, *where the order of f,* order f, *is the supremum of the number of points of $f^{-1}(x), x \in R$.*

Theorem 6. *If a non-empty topological space R has a directed family of locally finite closed coverings of order $\leq n+1$ which follows out the topology of R fully, then there exist a paracompact Hausdorff space A with* $\operatorname{Ind} A = 0$ *and a closed continuous mapping f of A onto R with* order $f \leq n+1$.

Theorem 7. *If there exists a closed continuous mapping f with* order $f \leq n+1$ *of a completely regular space A with* $\operatorname{ind} A = 0$ *onto a topological space R, then R has a directed family of finite closed coverings of order $\leq n+1$ which follows out the topology of R locally.*

Corollary 1. *Let A be a completely regular space with* $\operatorname{ind} A = 0$ *and R a topological space. If there exists a closed continuous mapping f of A onto R such that* order $f \leq n+1$, *then R is a regular space with* $\operatorname{ind} R \leq n$.

Theorem 8. *If there exists a closed continuous mapping f with* order $f \leq n+1$ *of a normal space* A *with* $\operatorname{Ind} A=0$ *onto a topological space* R, *then* R *has a directed family of finite closed coverings of order* $\leq n+1$ *which follows out the topology of* R *globally.*

Corollary 2. *Let* A *be a normal space with* $\operatorname{Ind} A=0$ *and* R *a topological space. If there exists a closed continuous mapping* f *of* A *onto* R *such that* order $f \leq n+1$, *then* R *is a normal space with* $\operatorname{Ind} R \leq n$.

Theorem 9. *If there exists a closed continuous mapping* f *with* order $f \leq n+1$ *of a paracompact Hausdorff space* A *with* $\operatorname{Ind} A=0$ *onto a topological space* R, *then* R *has a directed family of locally finite closed coverings of order* $\leq n+1$ *which follows out the topology of* R *fully.*

Corollary 3. *Let* A *be a paracompact Hausdorff space with* $\operatorname{Ind} A=0$ *and* R *a topological space. If there exists a closed continuous mapping* f *of* A *onto* R *such that* order $f \leq n+1$, *then* R *is a paracompact Hausdorff space with* $\operatorname{Ind} R \leq n$.